Abstract

Limits of Asymptotically Fuchsian Surfaces in a Closed Hyperbolic 3-Manifold

Fernando Al Assal

2023

Let M be a closed hyperbolic 3-manifold. Let $\nu_{\mathrm{Gr}\, M}$ denote the probability volume (Haar) measure of the 2-plane Grassmann bundle $\mathrm{Gr}\, M$ of M and let ν_T denote the area measure on $\mathrm{Gr}\, M$ of an immersed closed totally geodesic surface $T \subset M$. We say a sequence of π_1-injective maps $f_i : S_i \to M$ of surfaces S_i is *asymptotically Fuchsian* if f_i is K_i-quasifuchsian with $K_i \to 1$ as $i \to \infty$. We show that the set of weak-* limits of the probability area measures induced on $\mathrm{Gr}\, M$ by asymptotically Fuchsian minimal or pleated maps $f_i : S_i \to M$ of closed connected surfaces S_i consists of all convex combinations of $\nu_{\mathrm{Gr}\, M}$ and the ν_T.

Limits of Asymptotically Fuchsian Surfaces in a Closed Hyperbolic 3-Manifold

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
Fernando Al Assal

Dissertation Director: Yair Minsky

May, 2023

Acknowledgements

First of all, I would like to thank my advisor Yair Minsky for being a generous, patient and insightful mentor. This document would not exist without his guidance.

I would like to thank my wife Natalie Rose for her companionship and love, as well as my friends and family for being a part of my life.

I would like to thank Sebastian Hurtado-Salazar and Hee Oh for agreeing to read this. I would also like to thank Rich Schwartz for suggesting the adverb "asymptotically" rather than "increasingly" for the title, as well as Jeremy Kahn for suggesting the proof of Theorem 5.1.1. Finally, I would like to thank Danny Calegari, Sami Douba, James Farre, Ben Lowe and Franco Vargas-Pallete for useful discussions and correspondence.

Lastly, I would like to thank the Atlantic Ocean for opening up, flooding the surrounding continents with molten basalt, which cooled down and two hundred million years later provided me with a space for contemplation and leisure at East Rock Park.

Contents

Acknowledgements **iii**

1 Introduction **1**

2 Geometric realizations of asymptotically Fuchsian surfaces **9**

2.1 Quasiconformal maps and quasifuchsian groups 11

2.2 The Schwarzian derivative and the Bers norm 14

2.3 Pleated surfaces and the convex core 14

2.4 Proving Theorem 2.0.2 15

2.5 Minimal surfaces 26

2.6 Proving Theorem 2.0.3 26

3 Building surfaces out of good pants **34**

3.1 Building blocks 34

3.2 Assembling the surface 38

4 Connected surfaces going through every good pants **44**

5 Barycenters of the good pants are equidistributed **50**

5.1 Outline of the proof 51

5.2 Left and right 53
5.3 Equidistribution of feet in $\operatorname{Fr} M$ 55
5.4 Approximate barycenters of pants 62
5.5 Equidistribution of approximate barycenters in $\operatorname{Fr} M$ 64
5.6 Conclusion: equidistribution of barycenters in $\operatorname{Fr} M$ 66

6 Equidistributing surfaces 67

7 Non-equidistributing surfaces 70

List of Figures

1.1 The universal covers of asymptotically Fuchsian pleated surfaces are not necessarily embedded in $\mathbf{H}^3$ and may develop wrinkles as above, so they are never C^1-close to a totally geodesic plane 2

2.1 Asymptotically Fuchsian pleated surfaces in $\mathbf{H}^3$ are not necessarily embedded and develop wrinkles so they are never C^1-close to a totally geodesic disc 10

2.2 Visual outline of the proof of Theorem 2.0.1. We will flow the universal covers $\widetilde{f_i^m}(\widetilde{S_i})$ and $\widetilde{f_i^p}(\widetilde{S_i})$ of the asymptotically Fuchsian minimal and pleated surfaces normally till they hit a component H_i^+ of the boundary of the convex core. We will argue this process has a uniformly small area distortion (away from the pleating lamination, in the pleated case). 11

2.3 A visualization of the map F_i^η, flowing normally from P_i^η till H_i^+. Lemma 2.4.2 below shows that these lines indeed do not meet for i large enough. . . 16

2.4 Lemma 2.4.2 says that the boxes made out of flowing S^η and T^η for time $t(\eta)$ never meet. 17

2.5 If θ_i was smaller or equal to $\tilde{\theta}_i = \cos^{-1}(\tanh \tau_i(p)/ \tanh \eta)$ as in the figure above, then the supporting plane to H_i^+ containing $F_i(p)$ would intersect Δ, in a violation of convexity. 20

2.6 Illustrating Proposition 2.6.2 28
2.7 The angle α_i' is defined in a similar way to α_i, except that it is opposite to an intrinsic geodesic of D_i of length η, rather than a geodesic of $\mathbf{H}^3$. 30

3.1 Short orthogeodesics and feet of a good pants. 35
3.2 A good gluing between π_1 and π_2. 37
3.3 The feet of the good pants with cuff γ are well distributed in $\mathrm{N}^1(\sqrt{\gamma})$. 39

4.1 Proposition 4.0.3 tells us we may connect two good curves γ_0 and γ_1 via a bridge of good pants $\pi_0, \dots, \pi_n$. As suggested in the picture, the gluings are not necessarily (ϵ, R)-good. 46
4.2 Proof of Lemma 4.0.4. On the left, we have the dual graph X to the pants decomposition P of S. On the right, we have the double cover $\hat{X}$. T_i^1 and T_i^2 are the lifts of T_i to $\hat{X}$, which are nonseparating in $\hat{X}$. 47
4.3 Regluing S_1 to S_2 with a good gluing and reducing the number of components of S. 48

5.1 Left: midpoints and barycenter of an ideal triangle. Right: one of the three framed barycenters of an ideal triangle. 51
5.2 Pleated structure of a pair of pants consisting of two ideal triangles. 52
5.3 The pants P_R divided into left and right hexagons, with its left and right feet. 54
5.4 Spinning the hexagons of P_R into two ideal triangles. Picture drawn by Natalie Rose Schwartz. 55
5.5 Lifting the pants π and its left foot to $\mathbf{H}^3$. 63

7.1 The family of surfaces $S_{\epsilon,R,\mathbf{d}}$, which can accumulate on the totally geodesic surfaces T by appropriately choosing the degrees d of their covers. 72

Chapter 1

Introduction

Let $M = \Gamma\backslash\mathbf{H}^3$ be a closed hyperbolic 3-manifold, where $\Gamma \leq \mathrm{PSL}_2\,\mathbf{C}$ is a cocompact lattice. We say a sequence of π_1-injective (essential) maps $f : S_i \to M$ of surfaces S_i is *asymptotically Fuchsian* if f_i is K_i-quasifuchsian with $K_i \to 1$ as $i \to \infty$. For an almost-everywhere differentiable map $f : S \to M$ of a surface into M, we let $\nu(f)$ denote the probability area measure induced by f on the oriented 2-plane Grassmann bundle $\mathrm{Gr}\,M$ of M. (Precisely, if we let $\overline{f} : S \to \mathrm{Gr}\,M$ be given by $\overline{f}(p) = (f(p), T_{f(p)}f(S))$, then $\nu(f)$ is the pushforward via $\overline{f}$ of the pullback via f of the volume measure of M, normalized to have mass 1.) We let $\mathscr{G}$ denote the set of immersed closed totally geodesic surfaces in M. For $T \in \mathscr{G}$, we let ν_T denote the area measure of T on $\mathrm{Gr}\,M$. We let $\nu_{\mathrm{Gr}\,M}$ denote the probability volume (Haar) measure of $\mathrm{Gr}\,M$. The main theorem of the article is

Theorem 1.0.1. *The set of weak-* limits of $\nu(f_i)$, where $f_i : S_i \to M$ are asymptotically Fuchsian minimal or pleated maps of closed connected surfaces, consists of all measures of the form*

$$\nu = \alpha_M \nu_{\mathrm{Gr}\,M} + \sum_{T\in\mathscr{G}} \alpha_T \nu_T$$

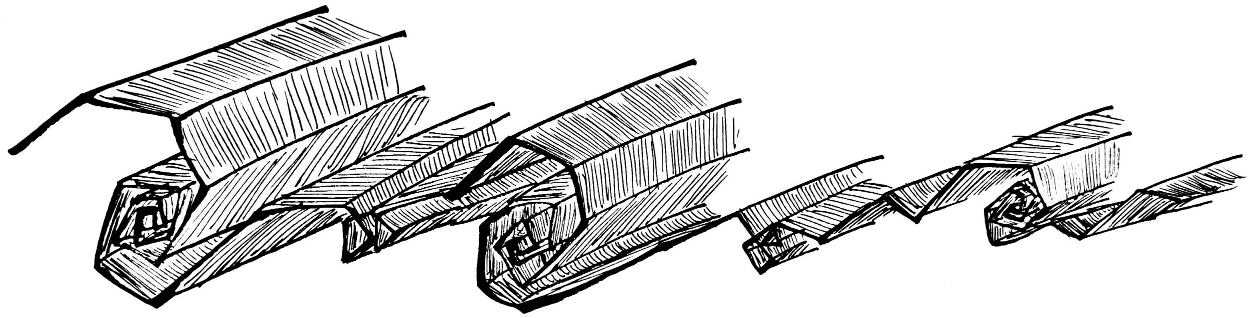

Figure 1.1: The universal covers of asymptotically Fuchsian pleated surfaces are not necessarily embedded in $\mathbf{H}^3$ and may develop wrinkles as above, so they are never C^1-close to a totally geodesic plane

where $\alpha_M + \sum_{T\in\mathscr{G}} \alpha_T = 1$.

An important part of the proof of Theorem 1.1 is showing that the weak-* limits of convergent subsequences of $\nu(f_i)$ do not depend on whether f_i is minimal or pleated, or in particular on the choice of pleated map. This is despite the fact that, in the pleated case, the universal covers of $f_i(S_i)$ do not converge to a geodesic plane in the C^1 sense.

Theorem 1.0.2. *Suppose $f_i : S_i \to M$ are essential asymptotically Fuchsian maps of a closed connected surface. Let f_i^p and f_i^m be, respectively, pleated and minimal maps homotopic to f_i. Then, the probability area measures $\nu(f_i^p)$ and $\nu(f_i^m)$ have the same weak-* limit along any convergent subsequence.*

Theorem 1.0.1 is in contrast with the case in which the maps $f_i : S_i \to M$ are all Fuchsian and the S_i are all distinct. Then, the surfaces $f_i(S_i)$ equidistribute in $\mathrm{Gr}\, M$, namely

Theorem (Shah-Mozes)**.** $\nu(f_i) \xrightarrow{\star} \nu_{\mathrm{Gr}\, M}$ *as* $i \to \infty$.

This follows from a more general theorem of Shah and Mozes ([22]). This is an article about unipotent dynamics, that builds on work of Dani, Margulis and Ratner. A special case of the main theorem in [22] is that a sequence of infinitely many distinct orbit closures of the unipotent flow in $\mathrm{Gr}\,M$ equidistributes. Due to Ratner ([18]), these orbit closures are either totally geodesic surfaces or all of $\mathrm{Gr}\,M$.

More recently, Margulis-Mohammadi ([17]) and Bader-Fisher-Miller-Stover ([1]) showed that if M contains infinitely many distinct totally geodesic surfaces, then M is aritmhetic. (On the other hand, it was already known, due to Reid ([19]) and Maclachlan-Reid ([16]) that if M is arithmetic, then it contains either zero or infinitely many distinct totally geodesic surfaces.) This rigid behavior of totally geodesic surfaces, however, is not shared by the nearly Fuchsian surfaces of M. Due to the surface subgroup theorem of Kahn and Markovic ([8]), any closed hyperbolic 3-manifold M has infinitely many essential K-quasifuchsian surfaces, for any $K > 1$.

The Kahn-Markovic construction of surface subgroups has a probabilistic flavor. The building blocks from which the nearly Fuchsian surfaces are assembled are the (ϵ, R)-good pants, which are the maps $f : P \to M$ from a pair of pants P taking the cuffs of P to (ϵ, R)-good curves in M – the closed geodesics with complex translation length 2ϵ-close to $2R$. We say two (ϵ, R)-good pants f and g are equivalent if f is homotopic to $g \circ \phi$, for some orientation-preserving homeomorphism $\phi : P \to P$. For more detailed and precise definitions, see Section 2.

A crucial reason why this construction works is that the good pants incident to a given good curve γ come from a well-distributed set of directions. Precisely, the *feet* of the good pants are well-distributed in the unit normal bundle $\mathrm{N}^1(\gamma)$ of γ. The feet of a good pants $\pi = f : P \to M$ are the derivatives of the unit speed geodesic segments connecting a cuff of $f(P)$ to another, meeting both cuffs orthogonally. Each cuff has two feet, and it turns out that they define the same point, the *foot*, denoted $\mathbf{ft}(\pi)$, in the quotient $\mathrm{N}^1(\sqrt{\gamma})$ of $\mathrm{N}^1(\gamma)$ by $n \mapsto n + \mathbf{hl}(\gamma)/2$, where $\mathbf{hl}(\gamma)$ is half of the translation length of γ.

The precise statement of the equidistribution of the feet follows below, from the article of Kahn and Wright [10] with proof in the supplement [11]. In [10] Kahn and Wright extend the surface subgroup theorem to the case where M has finite volume, while simplifying some elements of the original proof of Kahn-Markovic. The proof of the well-distribution of feet in [11] follows a different approach than the original Kahn-Markovic argument in [8]. In the latter, the pants are constructed by flowing tripods via the frame flow. In the former, pants with a given cuff are constructed from geodesic segments meeting the cuff orthogonally (the *orthogeodesic connections*). Denote the space of (ϵ, R)-good curves in M as $\mathbf{\Gamma}_{\epsilon,R}$ and the space of (ϵ, R)-good pants having γ as a cuff as $\mathbf{\Pi}_{\epsilon,R}(\gamma)$.

Theorem 1.0.3 (Kahn-Wright: Equidistribution of feet)**.** *There is $q = q(M) > 0$ so that if $\epsilon > 0$ is small enough and $R > R_0(\epsilon)$, the following holds. Let $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$. If $B \subset \mathrm{N}^1(\sqrt{\gamma})$, then*

$$(1-\delta)\lambda(N_{-\delta}B) \leq \frac{\#\{\pi \in \mathbf{\Pi}_{\epsilon,R}(\gamma) : \mathbf{ft}_\gamma\, \pi \in B\}}{C_{\epsilon,R,\gamma}} \leq (1+\delta)\lambda(N_\delta B),$$

where $\lambda = \lambda_\gamma$ is the probability Lebesgue measure on $\mathrm{N}^1(\sqrt{\gamma})$, $\delta = e^{-qR}$, $N_\delta(B)$ is the δ-neighborhood of B, $N_{-\delta}(B)$ is the complement of $N_\delta(\mathrm{N}^1(\sqrt{\gamma}) - B)$ and $C_{\epsilon,R,\gamma}$ is a constant depending only on ϵ, R and $\mathbf{l}(\gamma)$.

This theorem will be used in many ways in the article. It implies that a nearly geodesic surface $S(\epsilon, R)$ may be built using one representative of each equivalence class of (ϵ, R)-good pants. The equidistribution of feet (in a slight generalization explained in Section 5) will also be used to show that these surfaces equidistribute in $\mathrm{Gr}\, M$ as $\epsilon \to 0$. It will also be important in the construction of *non-equidistributing* asymptotically Fuchsian surfaces.

The surface $S(\epsilon, R)$ built out of a representative of each equivalence class may not be connected, however. And we do need, for our main theorem, a *connected* surface that goes through every good pants, meeting every cuff in a well-distributed set of directions. This can be achieved using the work of Liu and Markovic from [14]. Using their ideas, we can reglue the pants used to build $N = N(\epsilon, R)$ copies of $S(\epsilon, R)$ and obtain a connected surface that goes through every cuff in many directions.

Further directions

One can ask the same questions for finite-volume hyperbolic 3-manifolds M. Crucially, Kahn and Wright in [10] extended the surface subgroup theorem of Kahn and Markovic to this context, simplifying and sharpening some proofs on the way. The tool we still do not have to execute our construction there is the existence of equidistributing *connected* π_1-injective, closed asymptotically Fuchsian surfaces in M. This would use the work of Sun [24] that generalizes ideas of Liu and Markovic from [14] to finite-volume 3-manifolds. Another difference in this setting is that $\Gamma \backslash G$ has more complicated closed orbits of the unipotent flow, namely the closed horospheres associated to the cusps of M. It is not clear whether connected asymptotically Fuchsian surfaces may accumulate there or not.

Another direction is to extend these results to other homogeneous spaces $\Gamma \backslash G$, where G is a semisimple Lie group and $\Gamma < G$ a cocompact lattice. It has been shown that Γ has many surface subgroups, in the style of the Kahn-Markovic theorem, when Γ is a uniform lattice in a rank one simple Lie group of noncompact type distinct from $\mathrm{SO}_{2m,1}$ by Hämenstadt in [5] and when Γ is a uniform lattice in a center-free complex semisimple Lie group by Kahn, Labourie and Mozes in [7]. In the latter article, the authors show that their surface groups are *K-Sullivan* for any $K > 1$, which is a generalization of K-quasifuchsian for the higher rank setting. Again, it would be necessary to extend ideas of Liu and Markovic from [14] to those settings. Moreover, when G is a higher rank Lie group, there are more kinds of closed unipotent orbits in the homogeneous space $\Gamma \backslash G$ in which a sequence of asymptotically Fuchsian (or K-Sullivan with $K \to 1$) could perhaps accumulate in.

Outline

The large-scale structure of the article is the following. In Chapter 2, we show that that the weak-* limits of the probability area measures of asymptotically Fuchsian surfaces in M is a convex combination of the volume measure of $\mathrm{Gr}\, M$ and the area measures supported on closed geodesic surfaces. This is one of the directions of Theorem 1.0.1. The other direction of this equality will be proved in Chapters 3, 4, 5, 6 and 7. Details follow below.

In Chapter 2, we prove Theorem 1.0.2. Namely, we describe how nearly Fuchsian surfaces may be realized geometrically inside M as pleated or as minimal surfaces. We argue that, as these surfaces $f_i : S_i \to M$ become closer to Fuchsian, the weak-* limits of their area measures in $\mathrm{Gr}\, M$ do not depend on the choice of geometric structure. We do this by mapping the universal covers of our surfaces to a component of the convex core of $\pi_1(f_i)(\pi_1(S_i))$ via the normal flow, and arguing that this map has small derivatives in most of its domain. This is despite the fact that the universal covers of the pleated surfaces do not converge to a geodesic disc in the C^1 sense. For the case of minimal surfaces, we use the fact that their principal curvatures are uniformly small, as shown by Seppi in [21].

Using a theorem of Lowe for minimal surfaces from [15], we conclude that the limiting measures are $\mathrm{PSL}_2\,\mathbf{R}$-invariant. Thus, due to the Ratner measure classification, they are a convex combination of the volume measure of $\mathrm{Gr}\, M$ and area measures of the totally geodesic surfaces of M. This shows one direction of Theorem 1.0.1.

In Chapter 3, we explain how to construct nearly geodesic closed essential surfaces in M, following Kahn, Markovic and Wright ([8] and [10]). We define their building blocks, the (ϵ, R)-good pants, and the correct ((ϵ, R)-*good*) way to glue them together so the result is nearly Fuchsian. Finally, we explain how to use the equidistribution of feet (Theorem 3.2.3), together with the Hall marriage theorem from combinatorics, to show that a copy of each good pants may be glued via good gluings to form a closed surface $S(\epsilon, R)$.

In Chapter 4, we follow ideas of Liu and Markovic in [14] to explain how to reassemble a *connected* nearly Fuchsian closed essential surface $\hat{S}(\epsilon, R)$ from $N = N(\epsilon, R)$ copies of the surface built in Chapter 2. In particular, this connected surface defines the same area measure $\nu(\epsilon, R)$ in $\mathrm{Gr}(M)$ as $S(\epsilon, R)$.

In Chapter 5, we endow the connected surface built out of the same number of copies of each good pants $\hat{S}(\epsilon, R)$ with the pleated structure in which every good pants is glued from two ideal triangles. We show that the barycenters of these triangles equidistribute in the frame bundle $\mathrm{Fr}\, M$ of M as the surfaces become more Fuchsian (namely, as $\epsilon \to 0$ and $R(\epsilon) \to \infty$). To do so, we use a generalization of the equidistribution of feet (Theorem 3.2.3), in which a continuous function $g \in C(\mathrm{N}^1(\sqrt{\gamma}))$ plays the role of the set B in the statement above. We also use the fact, from a version due to Lalley in [13], that asymptotically almost surely, the cuffs of the pants equidistribute in the unit tangent bundle $\mathrm{T}^1 M$.

In Chapter 6, from the equidistribution of the barycenters of the triangles, we conclude that the surfaces $\hat{S}(\epsilon, R)$ built from the triangles equidistribute as $\epsilon \to 0$ and $R(\epsilon) \to \infty$. This is because each triangle can be obtained from the right action of a subset $\Delta \subset \mathrm{PSL}_2\, \mathbf{R}$ on the barycenter. The approach we take in Chapters 5 and 6 is similar to the one used by Labourie in [12] to show that certain perhaps disconnected asymptotically Fuchsian surfaces equidistribute in M. A difference is that the surfaces in [12] are built from a different multiset of good pants that comes from the original Kahn-Markovic construction. It is not clear, for example, how many copies of each pants are used to build those asymptotically Fuchsian surfaces.

In Chapter 7, we build a family of nearly Fuchsian surfaces by gluing the equidistributing surfaces $\hat{S}(\epsilon, R)$ of Chapters 5 and 6 to high degree covers of totally geodesic surfaces in M. To do so, we need the fact that a high degree cover of the totally geodesic surfaces of M may be built from good gluings of good pants. This was shown by Kahn and Markovic in [9] in order to prove the Ehrenpreis conjecture. We show that as these hybrid surfaces become asymptotically Fuchsian, they may accumulate on any of the totally geodesic surfaces.

Chapter 2

Geometric realizations of asymptotically Fuchsian surfaces

Suppose $f : S \to M$ is an essential nearly Fuchsian immersion of a closed connected orientable surface. Then, f is homotopic to maps with interesting geometric properties, namely a unique minimal map and many pleated maps. In this chapter, we will describe these geometric realizations and show that their area measures in $\operatorname{Gr} M$ have the same limit as they become asymptotically Fuchsian.

Precisely, suppose $f_i : S_i \to M$ are asymptotically Fuchsian maps of closed connected surfaces S_i. Let f_i^p and f_i^m be, respectively, pleated and minimal maps homotopic to f_i. Let $\nu(f_i^p) = p_i$ and $\nu(f_i^m) = m_i$ be the probability area measures induced by these maps on the 2-plane Grassmann bundle $\operatorname{Gr} M$. The main theorem of this chapter is the following, which was labeled as Theorem 1.2 in the introduction.

Theorem 2.0.1. *A subsequence m_{i_j} satisfies $m_{i_j} \xrightarrow{\star} \nu$ as $j \to \infty$ if and only if $p_{i_j} \xrightarrow{\star} \nu$.*

Let $\hat{m}_i = \hat{\nu}(f_i^m)$ be the probability measure induced by f_i^m on the frame bundle $\operatorname{Fr} M$. By the weak-* compactness of the probability measures on $\operatorname{Fr} M$, the $\hat{m}_i$ converge to a measure

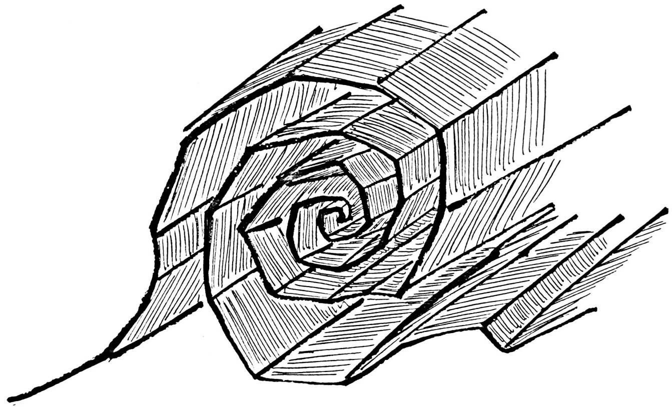

Figure 2.1: Asymptotically Fuchsian pleated surfaces in $\mathbf{H}^3$ are not necessarily embedded and develop wrinkles so they are never C^1-close to a totally geodesic disc

$\hat{\nu}$ along a subsequence. As shown by Lowe in Proposition 5.2 of [15] and Labourie in Section 5 of [12], the measure $\hat{\nu}$ is invariant under the right action of $\mathrm{PSL}_2\,\mathbf{R}$. Thus, from the Ratner measure classification theorem [18], it follows that the weak-* subsequential limits of m_i are of the form

$$\nu = \alpha_M \nu_{\mathrm{Gr}\,M} + \sum_{T \in \mathscr{G}} \alpha_T \nu_T. \tag{$\star$}$$

As before, $\mathscr{G}$ is a set containing a representative of each commensurability class of closed immersed totally geodesic surfaces in M, $\nu_{\mathrm{Gr}\,M}$ is the probability Haar measure on $\mathrm{Gr}\,M$, and ν_T is the probability area measure of an immersed closed totally geodesic surface $T \subset M$. The coefficients α_M and α_T sum to 1.

This, combined with Theorem 2.0.1, shows one of the directions of the main theorem of the article, Theorem 1.0.1. In Chapters 5, 6 and 7, we will show that given any ν of the form ($\star$), we may find asymptotically Fuchsian connected closed surfaces in M with limiting measure ν.

Let H_i^+ be a (the top) component of the boundary of the convex core of $Q_i = \pi_1(f)(\pi_1 S_i)$. Let f_i^h be the pleated map homotopic to f_i that whose lift to the universal cover maps $\hat{S}_i$

Figure 2.2: Visual outline of the proof of Theorem 2.0.1. We will flow the universal covers $\widetilde{f_i^m}(\widetilde{S_i})$ and $\widetilde{f_i^p}(\widetilde{S_i})$ of the asymptotically Fuchsian minimal and pleated surfaces normally till they hit a component H_i^+ of the boundary of the convex core. We will argue this process has a uniformly small area distortion (away from the pleating lamination, in the pleated case).

into H_i^+ and say $h_i = \nu(f_i^h)$. To prove Theorem 4.1, we show that each of p_i and m_i has the same weak-* subsequential limits as h_i.

Theorem 2.0.2. *A subsequence p_{i_j} satisfies $p_{i_j} \xrightharpoonup{\star} \nu$ as $j \to \infty$ if and only if $h_{i_j} \xrightharpoonup{\star} \nu$.*

Theorem 2.0.3. *A subsequence m_{i_j} satisfies $m_{i_j} \xrightharpoonup{\star} \nu$ as $j \to \infty$ if and only if $h_{i_j} \xrightharpoonup{\star} \nu$.*

Theorems 2.0.2 and 2.0.3 are in turn proven by flowing the universal covers $\widetilde{f_i^m}(\widetilde{S_i})$ and $\widetilde{f_i^p}(\widetilde{S_i})$ normally into H_i^+. We argue that this process has uniformly small area distortion. In the pleated case of Theorem 2.0.2, we need to argue a definite distance $\eta > 0$ away from the bending lamination of $\widetilde{f_i^p}(\widetilde{S_i})$ to avoid complicated wrinkles as in Figure 2.1. Then, we take $\eta \to 0$. In the minimal case of Theorem 2.0.3, we use the result of Seppi [21] that says that the principal curvatures of $\widetilde{f_i^m}(\widetilde{S_i})$ go uniformly to zero as the quasiconformal constant K_i tends to 1.

2.1 Quasiconformal maps and quasifuchsian groups

Let $\Omega \subset \hat{\mathbf{C}}$ be a domain. A continuous map $h : \Omega \to \hat{\mathbf{C}}$ is quasiconformal if its weak derivatives are locally in $L^2(\Omega)$ and it satisfies the Beltrami equation

$$\partial_z h(z) = \mu(z)\partial_{\bar{z}} h(z)$$

for almost every $z \in \Omega$ for some $\mu \in L^\infty(\Omega)$ with $||\mu||_{L^\infty(\Omega)} < 1$. The derivatives $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$ are understood in the distributional sense.

We say that $h : \Omega \to \hat{\mathbf{C}}$ is K-quasiconformal if μ, which is called the Beltrami differential of h, satisfies

$$K(h) := \frac{1 + ||\mu||_\infty}{1 - ||\mu||_\infty} \leq K.$$

In general, μ is a Beltrami differential in a domain $\Omega \subset \hat{\mathbf{C}}$ if it is an element of the open unit ball around the origin $B_1(0)$ of $L^\infty(\Omega)$. The measurable Riemann mapping theorem says that given a Beltrami differential in $\hat{\mathbf{C}}$, we may find a unique quasiconformal mapping $h : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ fixing 0, 1 and ∞ with $\partial_z h = \mu \partial_{\bar{z}} h$.

Quasiconformal maps enjoy the following compactness property that will be useful to us. (It is Lemma 6 on page 21 of [4].)

Lemma (Compactness)**.** *Let $h_i : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a sequence of K-quasiconformal maps fixing* 0, 1 *and* ∞. *Then the h_i converge uniformly to h as $i \to \infty$, where h is a K-quasiconformal map.*

It turns out that 1-quasiconformal maps are conformal, which is a regularity theorem for the solutions of the Beltrami equation. Thus, it follows that if the h_i are K_i-quasiconformal fixing 0, 1 and ∞ with $K_i \to 1$ as $i \to \infty$, then they converge uniformly to the identity.

Let $\mathbf{U} \subset \hat{\mathbf{C}}$ denote the upper half plane, and let $\mathbf{L} = \hat{\mathbf{C}} \setminus \bar{\mathbf{U}}$. We define the universal Teichmüller space of $\mathbf{U}$ as

$$\mathscr{T}(\mathbf{U}) = \{h : \hat{\mathbf{C}} \to \hat{\mathbf{C}} \text{ quasiconformal fixing } 0, 1 \text{ and } \infty \ : \ h|_{\mathbf{L}} \text{ is conformal}\}.$$

To obtain elements of $\mathscr{T}(\mathbf{U})$, let μ be a Beltrami differential in $\mathbf{U}$. We may extend it to a Beltrami differential also denoted μ in $\hat{\mathbf{C}}$ by setting $\mu|_{\mathbf{L}} = 0$. By the measurable Riemann mapping theorem, there is a unique quasiconformal mapping h of $\hat{\mathbf{C}}$ that fixes 0, 1 and ∞ and satisfies $\partial_z h = \mu \partial_{\bar{z}} h$. Moreover, $\partial_{\bar{z}} h = 0$ in $\mathbf{L}$, so $h|_{\mathbf{L}}$ is conformal.

A Jordan curve $\Lambda \subset \hat{\mathbf{C}}$ is a K-quasicircle if

$$K = \inf\{K(h) : h \in \mathscr{T}(\mathbf{U}) \text{ and } \Lambda = h(\partial\mathbf{U})\}.$$

Note that this infimum is achieved: if h_i are elements of $\mathscr{T}(\mathbf{U})$ with $K(h_i) \to K$, then by the compactness lemma, the h_i converge uniformly to a K-quasiconformal mapping of $\hat{\mathbf{C}}$ fixing 0, 1 and ∞ with $\Lambda = h(\partial\mathbf{U})$.

A group $Q \leq \mathrm{PSL}_2\,\mathbf{C}$ is *K-quasifuchsian* if $F = hQh^{-1}$ is a Fuchsian group for some K-quasiconformal map $h : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$. Up to conjugating Q by a $g \in \mathrm{PSL}_2\,\mathbf{C}$, we can say that its limit set Λ_Q contains 0, 1 and ∞. Thus, there is a K-quasiconformal mapping $h \in \mathscr{T}(\mathbf{U})$ so that $\Lambda_Q = f(\partial\mathbf{U})$. In particular, we see that Λ_Q is a *K-quasicircle* – the image of a circle under a K-quasiconformal map. These are nowhere differentiable Hölder curves.

A continuous, π_1-injective map $f : S \to M$ of a hyperbolic surface S into a hyperbolic 3-manifold M is *K-quasifuchsian* if $f_*(\pi_1 S) \leq \Gamma \cong \pi_1 M \leq \mathrm{PSL}_2\,\mathbf{C}$ is a K-quasifuchsian group. Given a K-quasifuchsian subgroup Q of the Kleinian group $\Gamma \cong \pi_1 M$, we may recover a K-quasifuchsian map $f : S \to M$ in the following way. As described above, Q gives rise to a K-quasiconformal map $h \in \mathscr{T}(\mathbf{U})$, whose restriction to $\partial\mathbf{U} \cong \partial_\infty \mathbf{H}^2$ may be extended to a Q-equivariant map $\tilde{f} : \mathbf{H}^2 \to \mathbf{H}^3$. The map $\tilde{f}$ in turn descends to $f : S \to M$. (We will describe examples of this extensions as minimal or pleated maps in detail below.)

A sequence of maps $f : S_i \to M$ of hyperbolic surfaces S_i into a hyperbolic 3-manifold is *asymptotically Fuchsian* if the f_i are K_i-quasifuchsian for $K_i \to 1$ as $i \to \infty$. Given such a sequence, we may find a sequence of K_i-quasiconformal maps $h_i \in \mathscr{T}(\mathbf{U})$ that conjugate $Q_i = (f_*)(\pi_1 S_i)$ into $\mathrm{PSL}_2\,\mathbf{R}$. From the compactness theorem of quasiconformal maps, it follows that the h_i converge uniformly to the identity. In particular, the limit sets Λ_{Q_i} are sandwiched between two circles at an Euclidean distance going to zero as $i \to \infty$.

2.2 The Schwarzian derivative and the Bers norm

The *Schwarzian derivative* of a holomorphic function f with nonvanishing derivative is given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

This vanishes precisely at the Möbius transformations and it can be shown that if f_i converges uniformly to a Möbius transformation as $i \to \infty$, then $S_{f_i} \to 0$ as $i \to \infty$.

The *Bers norm* of $f \in \mathscr{T}(\mathbf{U})$ is given by

$$\|f\|_B := \sup_{z \in \mathbf{L}} |S_f(z)|\rho^2(z),$$

where ρ is the Poincaré metric of curvature -1 on $\mathbf{L}$. As the quasiconformal constant of f goes to 1, f converges uniformly to the identity on $\hat{\mathbf{C}}$, and so $\|f\|_B \to 0$.

2.3 Pleated surfaces and the convex core

A π_1-injective isometric map $f : S \to M$ of a surface S is *pleated* or *uncrumpled* if every $p \in S$ is inside a geodesic arc of S that is mapped to a geodesic arc of M. It turns out (see Proposition 8.8.2 of [25]) that the set $\lambda \subset S$ of points that lie in a single geodesic segment that gets mapped to a geodesic is a lamination on S, and that f is totally geodesic outside λ. The lamination λ is called the *pleating* or *bending* lamination, and can be given a transverse measure that keeps track of the bending angles between the totally geodesic pieces of $f(S)$.

A K-quasifuchsian map $f : S \to M$ is homotopic to many pleated surfaces – given any geodesic lamination $\lambda \subset S$, it is possible to find a pleated map homotopic to f whose pleating locus is λ. One such pleated map of note comes from the boundary of the convex core of the quasifuchsian group $Q = f_*(\pi_1 S)$. Let Λ be the limit set of Q. The convex core of

Q is the smallest set $\operatorname{core} Q \subset \mathbf{H}^3$ containing the geodesics with endpoints in Λ. Thurston showed that its boundary $\partial \operatorname{core} Q$ has two components H^- and H^+ that are the image of $\mathbf{H}^2$ under a Q-equivariant pleated map [3]. In particular, $f : S \to M$ is homotopic to a pleated map $f^h : S \to M$ so that $\widetilde{f^h}(\tilde{S}) = H^+$.

The pleated discs H^- and H^+ inherit an orientation from f, and in particular normal vector fields n^- and n^+ away from their bending loci. We will follow the convention that H^- is the component so that the trajectory from flowing a vector n^- via the geodesic flow will meet H^+ at some positive time.

Another pleated map homotopic to $f : S \to M$ of importance in this article is the one where the bending lamination consists of a pants decomposition of S as well as three spiraling geodesics per pants that divide the pants into two ideal triangles. We will keep track of these triangles to show that the surface built out of one copy of each (ϵ, R)-good pants equidistributes as $\epsilon \to 0$ in Chapter 5.

2.4 Proving Theorem 2.0.2

We are now ready to restate and prove Theorem 2.0.2. Let $f_i : S_i \to M$ be asymptotically Fuchsian maps, with $Q_i = (f_i)_*(\pi_1 S_i)$. Let H_i^- and H_i^+ be the components of $\partial \operatorname{core} Q_i$ (again, chosen so flowing normally from H_i^- gets you to H_i^+). Let f_i^p and f_i^h be pleated maps homotopic to f_i, where f_i^h has a lift to the universal cover $\widetilde{f_i^h} : \widetilde{S_i} \to \mathbf{H}^3$ so that $\widetilde{f_i^h}(\widetilde{S_i}) = H_i^+$. Let $p_i = \nu(f_i^p)$ and $h_i = \nu(f_i^h)$ be the area measures induced on $\operatorname{Gr} M$ by f_i^p and f_i^h, respectively.

Theorem. *A subsequence p_{i_j} satisfies $p_{i_j} \xrightarrow{\star} \nu$ as $j \to \infty$ if and only if $h_{i_j} \xrightarrow{\star} \nu$.*

Let $\Lambda_i \subset \hat{\mathbf{C}}$ be the limit set of Q_i Let $\widetilde{f_i^p} : \widetilde{S_i} \to \mathbf{H}^3$ be the lift of f_i^p to the universal cover so $\partial_\infty \widetilde{f_i^p}(\widetilde{S_i}) = \Lambda_i$. We define $P_i := \widetilde{f_i^p}(\widetilde{S_i})$. We let $\tilde{p}_i$ and $\tilde{h}_i$ be, respectively, the area measures

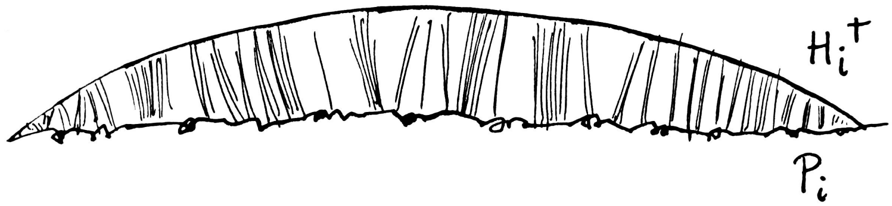

Figure 2.3: A visualization of the map F_i^η, flowing normally from P_i^η till H_i^+. Lemma 2.4.2 below shows that these lines indeed do not meet for i large enough.

induced by $\widetilde{f_i^p}$ and $\widetilde{f_i^h}$ on $\operatorname{Gr}\mathbf{H}^3$. We denote the pleating laminations of P_i and H_i^+ by λ_i and β_i, respectively. Finally, we define $\Sigma_i := \Gamma\backslash P_i$ and $R_i := \Gamma\backslash H_i^+$.

We let $n_t : \operatorname{Gr}\mathbf{H}^3 \to \mathbf{H}^3$ be the map taking $(p, P) \in \operatorname{Gr}\mathbf{H}^3$ to the point $q \in \mathbf{H}^3$ obtained by flowing p in the direction normal to P (from the orientation of P) for time t via the geodesic flow.

We define a map

$$F_i^\eta : P_i^\eta \longrightarrow H_i^+$$

by flowing $p \in P_i^\eta$ normally for the time $\tau_i(p)$ it takes to hit H_i^+. In other words, $F_i^\eta(p) = n_{\tau_i(p)}$.

We also let $\det(dF_i^\eta)$ be the Radon-Nikodym derivative

$$\det(dF_i^\eta) := \frac{d(F_i^\eta)^*\tilde{h}_i}{d\tilde{p}_i},$$

which is defined due to parts i and ii of the Propositon 2.4.1 below.

Proposition 2.4.1. *For $i \geq I_0(\eta)$, these maps F_i^η satisfy*

i. *F_i is differentiable outside of $(F_i^\eta)^{-1}(\beta_i) \cup \lambda_i$,*

ii. $\tilde{p}_i\left((F_i^\eta)^{-1}(\beta_i)\right) = 0$,

iii. $\|\det(dF_i^\eta) - 1\|_{L^\infty(P_i^\eta)} = o_i(1)$.

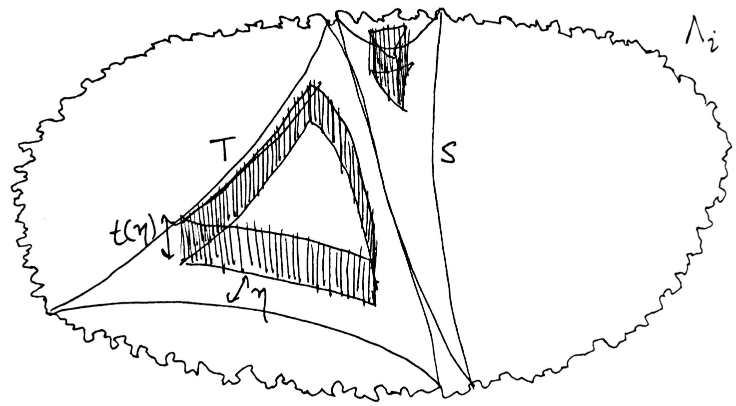

Figure 2.4: Lemma 2.4.2 says that the boxes made out of flowing S^η and T^η for time $t(\eta)$ never meet.

Proof. (We will drop the superscript η when convenient.)

i. Let $p \in P_i^\eta \setminus (F^{-1}(\beta_i) \cup \lambda_i)$. Then, F_i maps a small disc around p to a piece of a totally geodesic plane in H_i^+ via the normal flow. This is a differentiable map.

ii. For this we need the following

Lemma 2.4.2. *Supose $i \geq I_0(\eta)$. Then, there is $t(\eta) > 0$ so that for any two ideal triangles S and T in P_i, we have*

$$n_{s_1}(S^\eta) \cap n_{s_2}(T^\eta) = \emptyset$$

for all $0 \leq s_1, s_2 \leq t(\eta)$.

Proof. Without loss of generality, up to conjugating everything by Möbius transformations, we may take $S = \Delta$.

Recall that $\widetilde{f_i^p} : \mathbf{H}^2 \longrightarrow \mathbf{H}^3$ is the pleated map so that $\widetilde{f_i^p}(\mathbf{H}^2) = P_i$. We know that $\partial_\infty p_i$ is the K_i-quasiconformal homeomorphism $h_i : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ fixing 0, 1 and ∞ so that $h_i(\hat{\mathbf{R}}) = \Lambda_i$ (where $K_i \to 1$ as $i \to \infty$). In particular, $\widetilde{f_i^p}$ is the identity on Δ. Moreover, as discussed previously, h_i converges uniformly to the identity map as $i \to \infty$. Denote this modulus of uniform convergence as ω_i.

Define the following closed intervals in $\hat{\mathbf{R}} \subset \hat{\mathbf{C}}$:

$$I^1 = [0,1], \quad I^2 = [1,\infty] \quad \text{and} \quad I^3 = [\infty, 0].$$

Let T be a triangle in $P_i \setminus \lambda_i$, distinct from Δ. Then, $\widetilde{f_i^p}_i^{-1}(T)$ and $\widetilde{f_i^p}^{-1}(\Delta) = \Delta$ are triangles in the ideal triangulation $\widetilde{f_i^p}^{-1}(\lambda_i)$ of $\mathbf{H}^2$. In particular, they do not intersect, so the vertices of $\widetilde{f_i^p}^{-1}(T)$ all lie in I^ℓ for some $\ell \in \{1,2,3\}$. Thus, as f_i is uniformly ω_i close to the identity, the vertices of T are contained in $N_{\omega_i}(I^\ell)$, the ω_i-neighborhood of I^ℓ in $\hat{\mathbf{C}}$.

As the vertices of T are trapped in a shrinking neighborhood of I^ℓ, we have that

$$T^\eta \cap N_\eta(\Delta^\eta) = \emptyset,$$

which in turn implies that

$$\sup_{p \in T^\eta} \operatorname{dist}_{\mathbf{H}^3}(p, \Delta^\eta) \geq \eta.$$

It follows that $n_{s_1}(\Delta^\eta) \cap n_{s_2}(T^\eta) = \emptyset$ for $0 \leq s_1, s_2 \leq \eta/2$. We conclude the theorem holds for $t(\eta) = \eta/2$.

$\square$

For i sufficiently large depending on η, the lemma allows us to define a map

$$G_i : E^\eta \longrightarrow P_i^\eta$$

which takes $q \in P_i^{\eta,t}$ back to the point $p \in P_i^\eta$ so that $g_t(p,n) = q$. This is well defined as the components of E^η given by normal flow starting at some triangle of P_i^η never intersect. The map G_i is smooth and its restriction to H_i^η is Lipschitz and equal to $(F_i^\eta)^{-1}$. In particular, it takes sets of measure zero to sets of measure zero.

iii. It suffices to show that $\det(dF_i^\eta)$ converges uniformly to 1 in the fixed triangle Δ^η, namely

Proposition 2.4.3. *$\|\det(dF_i) - 1\|_{L^\infty(\Delta^\eta)} \to 0$ as $i \to \infty$.*

Indeed, if $T_{i_j} \subset P_i$ is a sequence of triangles, there are Möbius transformations f_{i_j} so that $f_{i_j} T_{i_j} (f_{i_j})^{-1} = \Delta$ while $f_{i_j} \Lambda_i f_{i_j}^{-1}$ is still trapped in a $\delta(i)$ neighborhood of $\mathbf{R}$, where $\delta(i) = o_i(1)$ and does not depend on j. Therefore, conjugating by f_{i_j} does not affect the following analysis and in particular $\det dF_i$ being uniformly close to 1 in Δ^η implies $\det dF_i$ is uniformly close to 1 in all of P_i^η.

Recall that τ_i is the time it takes for a point $p \in \Delta^\eta$ to hit H_i^+ via the normal flow, i.e.,

$$\tau_i(x) = \inf\{t > 0 : n_t(x) \in H_i^+\}.$$

In order to prove Proposition 2.4.3, we will need the following

Lemma 2.4.4. *$\|\tau_i\|_{C^1(\Delta^\eta)} \to 0$ as $i \to \infty$.*

Proof. We begin by showing that

$$\|\tau_i\|_{L^\infty(\Delta^\eta)} \to 0 \text{ as } i \to \infty. \tag{2.4.5}$$

Recall that $\Lambda_i = f_i(\mathbf{R})$, where the f_i are K_i-quasiconformal maps with $K_i \to 1$ that converge uniformly to the identity on $\hat{\mathbf{C}}$. In particular, we can find a function $\delta(i) \to 0$ with $i \to \infty$ so that $\Lambda_i \subset N_{\delta(i)}(\mathbf{R})$. Let Π_i^+ and Π_i^- be the totally geodesic planes satisfying

$$\partial_\infty \Pi_i^+ \cup \partial_\infty \Pi_i^- = \partial N_{\delta(i)}(\mathbf{R}),$$

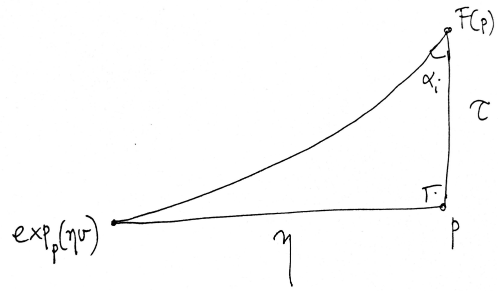

Figure 2.5: If θ_i was smaller or equal to $\tilde{\theta}_i = \cos^{-1}(\tanh \tau_i(p)/\tanh \eta)$ as in the figure above, then the supporting plane to H_i^+ containing $F_i(p)$ would intersect Δ, in a violation of convexity.

with Π_i^+ in the same side of the plane containing Δ as H_i^+. Let $T_i(x)$ be the time it takes for a point x to hit Π_i^+ via the normal flow, i.e.,

$$T_i(x) = \inf\{t > 0 : n_t(x) \in \Pi_i^+\}.$$

By construction, $\tau_i(x) \leq T_i(x)$ for $x \in \Delta^\eta$. In addition, as $\delta(i) \to 0$, we also have that $T_i(x) \to 0$ uniformly in x, with $i \to \infty$. This shows 2.4.5.

Let $p \in \Delta^\eta$ be a point outside of $F_i^{-1}(\beta_i)$ and let $v \in T_p\Delta^\eta$ be a unit vector. Now, we show that

$$d\tau_i(p)(v) \overset{i\to\infty}{\longrightarrow} 0 \tag{2.4.6}$$

uniformly in $(p, v) \in \mathrm{T}^1\,\Delta^\eta$.

Let $\theta_i(p, v)$ be the angle in $(0, \pi/2]$ that the geodesic normal to Δ through p makes with the curve $s \mapsto F_i(\exp_p sv)$, for $s \geq 0$. It suffices to show that this angle is uniformly close to $\pi/2$ as (p, v) varies in $\mathrm{T}^1\,\Delta^\eta$.

Let α_i be the angle based at $F_i(p)$ between the normal geodesic $t \mapsto n_t(p)$ to Δ at p and the geodesic segment from $F_i(p)$ to $\exp_p(\eta v)$. (See Figure 2.5.)

Note that $\alpha_i < \theta_i$. If that was not the case and θ_i was any smaller, the supporting plane of H_i^+ containing $F_i(p)$ would intersect the intrinsic disc $B_\eta(p) \subset \Delta$ of radius η, in a contradiction of convexity.

On the other hand, from trigonometry,

$$\cos\theta_i < \cos\alpha_i = \frac{\tanh\tau_i(p)}{\tanh\eta}.$$

Thus, as $\|\tau_i\|_{L^\infty(\Delta^\eta)} \to 0$ as $i \to \infty$, it follows that $\cos\theta_i \to 0$ uniformly in (p, v). This finishes the argument. □

Proof of proposition. For $p \in \mathbf{H}^3$, let $v, w \in \mathrm{T}_p\,\mathbf{H}^3$. We let $\mathrm{area}_p(v, w)$ denote the area spanned by v and w in $\mathrm{T}_p\,\mathbf{H}^3$, with respect to the hyperbolic metric $G = \langle\cdot,\cdot\rangle$ of $\mathbf{H}^3$. In formulas,

$$\mathrm{area}_p(v, w) = \left|\det\begin{bmatrix}\langle v, v\rangle & \langle v, w\rangle \\ \langle v, w\rangle & \langle w, w\rangle\end{bmatrix}\right|^{1/2}.$$

The Radon-Nikodym derivative $\det dF_i$ measures the area distortion caused by F_i. If $p \in \Delta_i^\eta$, we have

$$\det dF_i(p) = \frac{\mathrm{area}(dF_i(p)v, dF_i(p)w)}{\mathrm{area}(v, w)},$$

where v and w are distinct vectors in $\mathrm{T}_p^1\,\mathbf{H}^3$.

Consider the coordinates in $\mathbf{H}^3$ given by $(x, y, t) = n_t(x, y)$, where we choose $(x, y) \in \mathbf{H}^2$ so that ∂_x and ∂_y form an orthonormal basis for $\mathrm{T}_p\,\mathbf{H}^2$, where $\mathbf{H}^2$ denotes the geodesic plane containing Δ. In these coordinates, the metric on $n_t(\mathbf{H}^2)$ is given by

$$G_t = \cosh^2 t\, g_{\mathbf{H}^2} + dt^2$$

, where $g_{\mathbf{H}^2}$ is the hyperbolic metric of $\mathbf{H}^2$. We also have

$$F_i(x, y, 0) = (x, y, \tau_i(x, y)),$$

and for $v \in T_p\mathbf{H}^2$,

$$dF_i(p)(v) = v + d\tau_i(p)v\,\partial_t.$$

With these explicit formulae for dF_i and G_t, we can calculate the area distortion $\det dF_i(p)$, which turns out to be

$$\begin{aligned}
\det dF_i(p) &= \frac{\operatorname{area}(dF_i(p)\,\partial_x, dF_i(p)\,\partial_y)}{\operatorname{area}(\partial_x, \partial_y)} \\
&= \left|\det \begin{bmatrix} \cosh^2\tau_i(p) + (\partial_x\tau_i(p))^2 & \partial_x\tau_i(p)\,\partial_y\tau_i(p) \\ \partial_x\tau_i(p)\,\partial_y\tau_i(p) & \cosh^2\tau_i(p) + (\partial_y\tau_i(p))^2 \end{bmatrix}\right|^{1/2} \\
&= \left(\cosh^4\tau_i(p) + |\nabla\tau_i(p)|^2\cosh^2\tau_i(p)\right)^{1/2}.
\end{aligned}$$

Using Lemma 2.4.4 above, we see that

$$\|\det dF_i - 1\|_{L^\infty(\Delta^\eta)} \to 0 \text{ as } i \to \infty.$$

$\square$

As argued above, it follows that

$$\|\det dF_i - 1\|_{L^\infty(P_i^\eta)} \to 0 \text{ as } i \to \infty.$$

This concludes the proof of item **iii** of Proposition 2.4.1. $\square$

Let $H_i^\eta := F_i(P_i^\eta)$ and let R_i^η be its projection to M. Let $\tilde{h}_i^\eta$ be the area measure of H_i^η and let h_i^η be the restriction of h_i (the probability area measure of $R_i = \Gamma\backslash H_i^+$) to R_i^η.

Corollary 2.4.7. *$h_i^\eta(M \setminus R_i^\eta) \to 0$ as $\eta \to 0$.*

Proof. Given an ideal triangle $T \subset P_i \setminus \lambda_i$, the area of $F_i^\eta(T^\eta)$ is larger than that of T^η. Since R_i and $\Sigma_i = \Gamma\backslash P_i$ have the same area (as they are pleated and homotopic to each other), the corollary follows from the fact that $p_i(M \setminus \Sigma_i^\eta) \to 0$ as $\eta \to 0$. $\square$

Claim 2.4.8. *Let $(g_\alpha) \subset C(\mathrm{Gr}\,\mathbf{H}^3)$ be a bounded and equicontinuous family of functions, namely*

i. $\sup_\alpha \|g_\alpha\|_\infty < \infty$

ii. There is a function $w : (0, \infty) \to \mathbf{R}$ satisfying $w(\delta) \to 0$ as $\delta \to 0$ such that

$$|g_\alpha(x) - g_\alpha(y)| \leq w\left(\mathrm{dist}_{\mathrm{Gr}\,\mathbf{H}^3}(x, y)\right).$$

Then,

$$\sup_\alpha \left| \int_{\mathrm{Gr}\,\mathbf{H}^3} g_\alpha \, d\tilde{p}_i^\eta - \int_{\mathrm{Gr}\,\mathbf{H}^3} g_\alpha \, d\tilde{h}_i^\eta \right| \to 0 \text{ as } i \to \infty.$$

Proof. Since $\det dF_i = d(F_i^* \tilde{h}_i^\eta)/d\tilde{p}_i^\eta$, we have

$$\int_{\mathrm{Gr}\,\mathbf{H}^3} g_\alpha \, d\tilde{h}_i^\eta = \int_{P_i^\eta} g_\alpha(F_i(x)) \, \det dF_i(x) \, d\tilde{p}_i^\eta(x).$$

Thus,

$$\left| \lim_{i\to\infty} \int_{\mathrm{Gr}\,\mathbf{H}^3} g_\alpha \, d\tilde{p}_i^\eta - \lim_{i\to\infty} \int_{\mathrm{Gr}\,\mathbf{H}^3} g_\alpha \, d\tilde{h}_i^\eta \right| \leq \int_{P_i^\eta} |g_\alpha \circ F_i| \, |\det(dF_i) - 1| \, d\tilde{h}_i^\eta + \int_{P_i^\eta} |g_\alpha \circ F_i - g_\alpha| \, d\tilde{p}_i^\eta$$

From the boundedness and equicontinuity of g_α and the fact that $\det dF_i$ converges uniformly to 1 (Claim 2.4.1), we see that the right hand side of this inequality goes to zero. $\square$

Claim 2.4.9. *Let $g \in C(\operatorname{Gr} M)$. Then,*

$$\lim_{i\to\infty} \int_{\operatorname{Gr} M} g\, dp_i^{\eta} = \lim_{i\to\infty} \int_{\operatorname{Gr} M} g\, dh_i^{\eta}.$$

Proof. It suffices to take a g supported in a small geodesic ball $B \subset \operatorname{Gr} M$. Let $\tilde{B}$ be a lift of this ball to $\operatorname{Gr} \mathbf{H}^3$. Then, there is $\tilde{g} \in C(\operatorname{Gr} \mathbf{H}^3)$ and a finite set $K_i \subset \Gamma$ so that

$$\int_{\operatorname{Gr} M} g\, dp_i^{\eta} = \frac{1}{\operatorname{area}(\Sigma_i)} \sum_{\gamma\in K_i} \int_{\operatorname{Gr} \mathbf{H}^3} \tilde{g} \circ \gamma\, d\tilde{p}_i^{\eta}$$

and similarly,

$$\int_{\operatorname{Gr} M} g\, dh_i^{\eta} = \frac{1}{\operatorname{area}(R_i)} \sum_{\gamma\in K_i} \int_{\operatorname{Gr} \mathbf{H}^3} \tilde{g} \circ \gamma\, d\tilde{h}_i^{\eta}.$$

Note that $(\tilde{g} \circ \gamma)_{\gamma\in\Gamma}$ is a bounded and equicontinuous family of functions in $C(\operatorname{Gr} M)$.

We claim moreover that the K_i can be chosen so that

$$\frac{\sup_i \# K_i}{\operatorname{area}(\Sigma_i)} < \infty.$$

Indeed, let $2B \subset \operatorname{Gr} M$ be a ball of twice the radius as B, centered at the same point. Then, $\# K_i$ is no larger than the number of connected components of $\Sigma_i \cap 2B$ that meet B. Each such component C satisfies $\operatorname{area}(C) \geq c(B)$, where $c(B)$ is a constant depending only on B. Thus, we have

$$\# K_i \cdot c(B) \leq \operatorname{area}(\Sigma_i),$$

which shows that $\# K_i / \operatorname{area}(\Sigma_i) \leq c(B)^{-1}$.

Using the fact that $\operatorname{area}(\Sigma_i) = \operatorname{area}(R_i)$, we estimate

$$\left|\int_{\operatorname{Gr} M} g\, dp_i^\eta - \int_{\operatorname{Gr} M} g\, dh_i^\eta\right| \leq \frac{1}{\operatorname{area}(\Sigma_i)} \sum_{\gamma\in K_i} \left|\int_{\operatorname{Gr}\mathbf{H}^3} \tilde{g}\circ\gamma\, d\tilde{p}_i^\eta - \int_{\operatorname{Gr}\mathbf{H}^3} \tilde{g}\circ\gamma\, d\tilde{h}_i^\eta\right|$$
$$\leq \frac{\# K_i}{\operatorname{area}(\Sigma_i)} \sup_{\gamma\in\Gamma} \left|\int_{\operatorname{Gr}\mathbf{H}^3} \tilde{g}\circ\gamma\, d\tilde{p}_i^\eta - \int_{\operatorname{Gr}\mathbf{H}^3} \tilde{g}\circ\gamma\, d\tilde{h}_i^\eta\right|.$$

The term above goes to zero as $i\to\infty$ due to the boundedness of $\# K_i/\operatorname{area}(\Sigma_i)$ and the boundedness and equicontinuity of $(\tilde{g}\circ\gamma)_{\gamma\in\Gamma}$. □

Finally, let $g \in C(\operatorname{Gr} M)$. Then,

$$\left|\int_{\operatorname{Gr} M} g\, dp_i - \int_{\operatorname{Gr} M} g\, dh_i\right|$$
$$\leq \left|\int_{\operatorname{Gr} M} g\, dp_i^\eta - \int_{\operatorname{Gr} M} g\, dh_i^\eta\right| + \int_{\operatorname{Gr} M} |g|\cdot 1_{P_i\setminus P_i^\eta}\, dp_i + \int_{\operatorname{Gr} M} |g|\cdot 1_{H_i^+\setminus H_i^\eta}\, dh_i$$
$$\leq \left|\int_{\operatorname{Gr} M} g\, dp_i^\eta - \int_{\operatorname{Gr} M} g\, dh_i^\eta\right| + \|g\|_{L^\infty(\operatorname{Gr} M)} \left(p_i^\eta(M-\Sigma_i^\eta) + h_i^\eta(M-H_i^\eta)\right).$$

Claim 2.4.9 implies that the first summand in the expression above goes to zero as $i\to\infty$. The second summand, in turn, goes to zero as $\eta\to 0$, from Corollary 2.4.7. Since η was arbitrary, we have shown

$$\left|\int_{\operatorname{Gr} M} g\, dp_i - \int_{\operatorname{Gr} M} g\, dh_i\right| \to 0 \text{ as } i\to\infty.$$

In particular, if a subsequence p_{i_j} converges to ν, then so does h_{i_j} and vice-versa. This completes the proof of Theorem 2.0.2.

2.5 Minimal surfaces

A map $f : S \to M$ of a surface S into M is *minimal* if the principal curvatures of $f(S)$ (a *minimal surface*) sum to zero at every point. These surfaces turn out to be locally area-minimizing.

Let $f : S \to M$ be a π_1-injective map of a hyperbolic surface S into M. Schoen-Yau [23] and Sacks-Uhlenbeck [20] show that f is homotopic to a minimal map f^m. In addition, Uhlenbeck shows that if the principal curvatures $\pm\lambda(p)$ of $f^m(S)$ satisfy $\lambda(p) \in (-1, 1)$ for every $p \in f^m(S)$, then f^m is quasifuchsian and it is the unique minimal map in its homotopy class. In addition, Seppi [21] shows that for a minimal K-quasifuchsian map $f^m : S \to M$ with K small enough,

Theorem 2.5.1 (Seppi)**.** *The principal curvatures $\pm\lambda$ of $f^m(S)$ satisfy*

$$||\lambda||_{L^\infty(f^m(S))} \leq C \log K,$$

for an universal constant C.

Combining these theorems, we see that if $f_i : S_i \to M$ are asymptotically Fuchsian maps, then for i large enough f_i is homotopic to a unique minimal map f_i^m. In addition, the principal curvatures of $f_i^m(S_i)$ go to zero uniformly as $i \to \infty$.

2.6 Proving Theorem 2.0.3

We will now restate and prove Theorem 2.0.3. As before, $f_i : S_i \to M$ are asymptotically Fuchsian maps and f_i^h is the pleated map homotopic to f_i coming from the top component H_i^+ of $Q_i = (f_i)_*(\pi_1 S_i)$. We let f_i^m be the minimal maps homotopic to f_i. We denote the probability area measure induced by f_i^m and f_i^h as $m_i = \nu(f_i^m)$ and $h_i = \nu(f_i^h)$.

Theorem. *A subsequence m_{i_j} satisfies $m_{i_j} \overset{\star}{\rightharpoonup} \nu$ if and only if $h_{i_j} \overset{\star}{\rightharpoonup} \nu$.*

We let $\widetilde{f_i^m}$ be the lift of f_i^m to $\mathbf{H}^2$ so that $\partial_\infty \widetilde{f_i^m}$ is the limit set Λ_i of Q_i. We let $\tilde{m}_i$ and $\tilde{h}_i$ be the area measures induced by $\widetilde{f_i^m}$ and $\widetilde{f_i^h}$ on $\operatorname{Gr} \mathbf{H}^3$. As before, β_i is the bending lamination of H_i^+, $R_i = \Gamma \backslash H_i^+$ and we put $N_i := \Gamma \backslash D_i = f_i^m(S_i)$.

As in the proof of Theorem 2.0.2, we define a map

$$F_i : D_i \longrightarrow H_i^+$$

where $F_i(p)$ is given by flowing p in the direction normal to D_i for the time $\tau_i(p)$ it takes to hit H_i^+. Concisely, $F_i(p) = n_{\tau_i(p)}(p)$.

Claim 2.6.1. *The map F_i satisfies the following properties:*

i. *F_i is differentiable outside of $F_i^{-1}(\beta_i)$*

ii. $\tilde{m}_i(F_i^{-1}(\beta_i)) = 0$

iii. $\| \det(dF_i) - 1 \|_{L^\infty(D_i)} \to 0$ *as* $i \to \infty$.

To prove this claim, we will use the following rephrasing of Proposition 4.1 of Seppi in [21]:

Proposition 2.6.2. *Suppose i is large enough that the uniformizing map f_i has Bers norm $\|f_i\|_B < 1/2$. Then, we may find surfaces D_i^- and D_i^+ that are equidistant from D_i so that the region between D_i^- and D_i^+ is convex and thus contains* $\operatorname{core} Q_i$.

Moreover, given $x \in D_i$, there is a geodesic segment α from D_i^- to D_i^+, meeting D_i and $D_i^\pm$ orthogonally, whose length satisfies

$$\ell(\alpha) \leq \operatorname{arctanh}(2\|f_i\|_B). \tag{2.6.3}$$

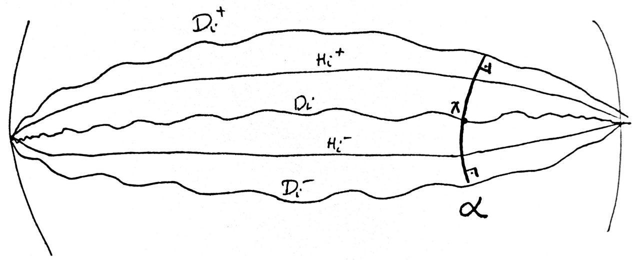

Figure 2.6: Illustrating Proposition 2.6.2

In particular, given $x_i \in D_i$, let P_i^+ and P_i^- be the geodesic planes tangent to D_i^+ and D_i^- at the endpoints of the segment α_i. From 2.6.3, we see that the distance between P_i^+ and P_i^- goes to zero as $i \to \infty$ and does not depend on the chosen point $x_i \in D_i$.

Proof of Claim 2.6.1. **i.** Let $x \in D_i \setminus F_i^{-1}(\beta_i)$. Then, F_i maps a disc around x to a piece of a totally geodesic plane in H_i^+ by the geodesic flow in the normal direction. This is a smooth map.

ii. Let E be the set containing all the points above D_i^- and below D_i^+. This set is foliated by surfaces D_i^t equidistant to D_i, for $t \in [-\operatorname{dist}(D_i^-, D_i), \operatorname{dist}(D_i, D_i^+)]$. The set E is also foliated by the orbits of the geodesic flow going through points in D_i and their normal vector. These flow lines never meet. If they did, the pullback metrics of the D_i^t on D_i would be degenerate, which is the not the case, as their principal curvatures are given by

$$\frac{\lambda - \tanh t}{1 - \lambda \tanh t} \quad \text{and} \quad \frac{-\lambda - \tanh t}{1 + \lambda \tanh t}$$

and $\lambda \in (-1, 1)$.

In particular, we can define a map

$$G_i : E \longrightarrow D_i,$$

which takes $y \in D_i^t \subset E$ back to the point $x \in D_i$ so that $g_t(x, n) = y$. This map is smooth and in particular, its restriction to H_i^+ is Lipschitz and hence takes sets of measure zero to sets of measure zero. Thus,

$$\tilde{m}_i(G_i(\beta_i)) = \tilde{m}_i(F_i^{-1}(\beta_i)) = 0.$$

iii. As before, first we show that the hitting times τ_i converge to zero uniformly on D_i in the C^1 sense.

Lemma 2.6.4. $\|\tau_i\|_{C^1(D_i)} \to 0$ *as* $i \to \infty$.

Proof. The fact that $\|\tau_i\|_{L^\infty(D_i)} \to 0$ as $i \to \infty$ follows readily from Seppi's Proposition 2.6.2. It remains to show that $d\tau_i(p)v \to 0$ as $i \to \infty$ uniformly in $\mathrm{T}^1(D_i - F_i^{-1}(\beta_i)$. This in turn will follow from Seppi's Theorem 2.5.1 that says that the principal curvatures of D_i converge uniformly from zero.

For $(p, v) \in \mathrm{T}^1 D_i$, we let $\theta_i(p, v)$ be the angle in $(0, \pi/2]$ that the geodesic normal to D_i at p makes with the curve $s \mapsto F_i(\exp_p sv)$. As in the pleated case (Lemma 2.4.4), it suffices to show that $\theta_i(p, v) \to \pi/2$ uniformly in $\mathrm{T}^1 D_i$.

Fix $\eta > 0$. Again, we let α_i be the angle based at $F_i(p)$ between the normal geodesic $t \mapsto n_t(p)$ to Δ at p and the geodesic segment from $F_i(p)$ to $\exp_p(\eta v)$. Let $\exp_p^{D_i} : \mathrm{T}_p^1 D_i \to D_i$ denote the exponential map intrinsic to D_i. We let α_i' be the angle based at $F_i(p)$ between the normal geodesic $t \mapsto n_t(p)$ to Δ at p and the *intrinsic* geodesic segment of D_i from $F_i(p)$ to $\exp_p^{D_i}(\eta v)$.

Note that $\alpha_i' < \theta_i$. If that was not the case, a supporting plane to H_i^+ based at $F_i(p)$ would meet D_i, in a violation of convexity.

Due to the principal curvatures of D_i going to zero as $i \to \infty$, it follows that the difference between α_i and α_i' also goes to zero unformly as $i \to \infty$. In other words, there is

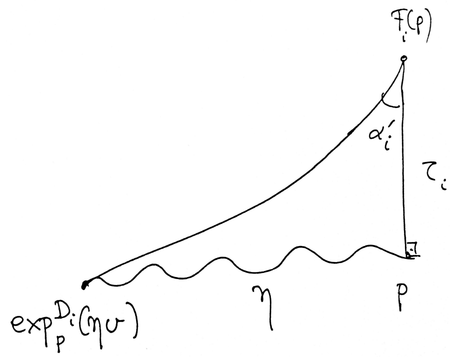

Figure 2.7: The angle α_i' is defined in a similar way to α_i, except that it is opposite to an intrinsic geodesic of D_i of length η, rather than a geodesic of $\mathbf{H}^3$.

a quantity $\omega_i \to 0$ as $i \to \infty$ (which depends on the choice of $\eta > 0$, but not of $(p, v) \in \mathrm{T}^1 D_i$), so that

$$|\cos \alpha_i - \cos \alpha_i'| \leq \omega_i.$$

But as before, $\cos \alpha_i = \tanh(\tau_i(p)) / \tanh \eta$. Thus,

$$\cos \theta_i \leq \frac{\tanh \|\tau_i\|_{L^\infty(D_i)}}{\tanh \eta} + \omega_i \xrightarrow{i \to \infty} 0.$$

$\square$

For each i, we choose coordinates on $\mathbf{H}^3$ given by $(x, y, t) = n_t(x, y)$, where (x, y) are coordinates for D_i chosen so that ∂_x and ∂_y form an orthonormal basis for $T_{p_i} D_i$. (The points p_i are chosen in the full measure set $D_i - F_i^{-1}(\beta_i)$.) In these coordinates, the metric G_t on $n_t(D_i)$ is given by

$$G_t = g_t + dt^2,$$

where at $n_t(p_i)$, the matrix entries of g_t corresponding to the basis ∂_x, ∂_y are given by

$$g_t = (\cosh t \, \mathrm{Id} + \sinh t A_i)^2 \tag{$\star$}$$

and A_i is the second fundamental form of D_i. (See Section 5 of Uhlenbeck [26] for details.)

As before, in these coordinates we have $F_i(x, y, 0) = (x, y, \tau_i(x, y))$ and so $dF_i(p)v = v + d\tau_i(p)v\,\partial/\partial t$. Thus,

$$\begin{aligned}
\det dF_i(p_i) &= \operatorname{area}(dF_i(p_i)\,\partial_x, dF_i(p_i)\,\partial_y) \\
&= \left|\det \begin{bmatrix} g_{\tau_i(p_i)}(\partial_x, \partial_x) + (\partial_x \tau_i(p_i))^2 & \partial_x \tau_i(p)\,\partial_y \tau_i(p_i) \\ \partial_x \tau_i(p_i)\,\partial_y \tau_i(p) & g_{\tau_i(p_i)}(\partial_y, \partial_y) + (\partial_y \tau_i(p_i))^2 \end{bmatrix}\right|^{1/2} \\
&= \left(|\partial_x|^2_{\tau_i(p_i)} |\partial_y|^2_{\tau_i(p_i)} + (\partial_x \tau_i(p_i))^2 |\partial_y|^2_{\tau_i(p_i)} + (\partial_y \tau_i(p_i))^2 |\partial_x|^2_{\tau_i(p_i)}\right)^{1/2},
\end{aligned}$$

Above, $|\partial_x|^2_{\tau_i(p_i)}$ and $|\partial_y|^2_{\tau_i(p_i)}$ denote, respectively, the first and second diagonal entries of $g_{\tau_i(p_i)}$. From Seppi's result (Theorem 2.5.1), we know that the second fundamental forms A_i converge uniformly to the zero matrix as $i \to \infty$. In view of the formula ($\star$), it follows that $|\partial_x|^2_{\tau_i(p_i)}$ and $|\partial_y|^2_{\tau_i(p_i)}$ both converge uniformly to 1 as $i \to \infty$.

In addition, from Lemma 2.6.4, we know that the derivatives of $\tau_i(p_i)$ converge to zero as $i \to \infty$. We can thus conclude that

$$\| \det dF_i(p) - 1 \|_{L^\infty(D_i)} \to 0 \text{ as } i \to \infty.$$

□

Let $\tilde{m}_i$ and $\tilde{h}_i$ be the area measures of D_i and H_i^+ in $\operatorname{Gr} \mathbf{H}^3$.

Claim 2.6.5. *Let $(g_\alpha) \subset C(\operatorname{Gr} \mathbf{H}^3)$ a bounded and equicontinuous family of functions. Then,*

$$\sup_\alpha \left| \int_{\operatorname{Gr} \mathbf{H}^3} g_\alpha \, d\tilde{m}_i - \int_{\operatorname{Gr} \mathbf{H}^3} g_\alpha \, d\tilde{h}_i \right| \to 0 \textit{ as } i \to \infty.$$

Proof. The proof is the same as the proof of Claim 2.4.8, substituting $\tilde{h}_i$ for $\tilde{h}_i^\eta$ and $\tilde{m}_i$ for $\tilde{p}_i^\eta$. □

Now, let $g \in C(\operatorname{Gr} M)$. In a similar fashion to the proof of Claim 2.4.9, we proceed to show that

$$\lim_{i\to\infty} \int_{\operatorname{Gr} M} g \, dm_i = \lim_{i\to\infty} \int_{\operatorname{Gr} M} g \, dh_i.$$

As before, we may choose g to be supported in a small geodesic ball $B \subset \operatorname{Gr} M$. For a lift $\tilde{B}$ of B to $\operatorname{Gr} \mathbf{H}^3$, there is $\tilde{g} \in C(\operatorname{Gr} \mathbf{H}^3)$ and a finite set $K_i \subset \Gamma$ so that

$$\int_{\operatorname{Gr} M} g \, dm_i = \frac{1}{\operatorname{area}(N_i)} \sum_{\gamma \in K_i} \int_{\operatorname{Gr} \mathbf{H}^3} \tilde{g} \circ \gamma \, d\tilde{m}_i$$

and similarly,

$$\int_{\operatorname{Gr} M} g \, dh_i = \frac{1}{\operatorname{area}(\Sigma_i)} \sum_{\gamma \in K_i} \int_{\operatorname{Gr} \mathbf{H}^3} \tilde{g} \circ \gamma \, d\tilde{h}_i.$$

As before, $(\tilde{g} \circ \gamma)_{\gamma \in \Gamma}$ is a bounded and equicontinuous family of functions in $C(\operatorname{Gr} M)$, and the K_i can be chosen so that $\sup_i \# K_i / \operatorname{area}(\Sigma_i) < \infty$.

Now we estimate

$$\begin{aligned}
&\left| \int_{\operatorname{Gr} M} g \, dm_i - \int_{\operatorname{Gr} M} g \, dh_i \right| \leq \frac{1}{\operatorname{area}(\Sigma_i)} \sum_{\gamma \in K_i} \left| \frac{\operatorname{area}(\Sigma_i)}{\operatorname{area}(N_i)} \int_{\operatorname{Gr} \mathbf{H}^3} \tilde{g} \circ \gamma \, d\tilde{m}_i - \int_{\operatorname{Gr} \mathbf{H}^3} \tilde{g} \circ \gamma \, d\tilde{h}_i \right| \\
&\leq \frac{\# K_i}{\operatorname{area}(\Sigma_i)} \sup_{\gamma \in \Gamma} \left[\left| \int_{\operatorname{Gr} \mathbf{H}^3} \tilde{g} \circ \gamma \, d\tilde{m}_i - \int_{\operatorname{Gr} \mathbf{H}^3} \tilde{g} \circ \gamma \, d\tilde{h}_i \right| + \left| 1 - \frac{\operatorname{area}(\Sigma_i)}{\operatorname{area}(N_i)} \right| \|\tilde{g} \circ \gamma\|_{L^1(\tilde{h}_i)} \right].
\end{aligned}$$

The upper bound above goes to zero as $i \to \infty$ due to the boundedness of $\# K_i / \operatorname{area}(\Sigma_i)$; the equicontinuity and boundedness of $(\tilde{g} \circ \gamma)_{\gamma \in \Gamma}$ and the fact that $\operatorname{area}(\Sigma_i)/\operatorname{area}(N_i)$ goes to 1.

To see the latter, say $\pm \lambda_i$ are the principal curvatures of N_i and g_i is the genus of S_i.

Using the Gauss-Bonnet formula, we have

$$\begin{aligned}\left|1 - \frac{\operatorname{area}(\Sigma_i)}{\operatorname{area}(N_i)}\right| &= \frac{1}{\operatorname{area}(N_i)}\left|\int_{N_i} 1\, d\,\mathrm{area} - 2\pi(2g_i - 2)\right| \\ &= \frac{1}{\operatorname{area}(N_i)}\left|\int_{N_i} 1\, d\,\mathrm{area} - \int_{N_i} \lambda_i^2\, d\,\mathrm{area}\right| \\ &\leq \frac{1}{\operatorname{area}(N_i)}\|1 - \lambda_i^2\|_{L^\infty(N_i)}\operatorname{area}(N_i).\end{aligned}$$

We know that $\|1 - \lambda_i^2\|_{L^\infty(N_i)} \to 0$ as $i \to \infty$ due to Seppi's Theorem 2.5.1.

Chapter 3

Building surfaces out of good pants

In this chapter, we will outline how to construct a π_1-injectively immersed closed oriented nearly Fuchsian surface in M out of good pants. This is the Kahn-Markovic surface subgroup theorem from [8], though our exposition will line up with that of Kahn and Wright in [10] as well as use some notation from Liu and Markovic in [14].

3.1 Building blocks

The following paragraphs define the many terms related to the building blocks of this construction.

An orthogeodesic γ between two closed geodesics $\alpha_0, \alpha_1 \subset M$ is a geodesic segment parametrized with unit speed going from α_0 to α_1 and meeting both curves orthogonally.

We denote by $\mathbf{\Gamma}_{\epsilon,R}$ the space of (ϵ, R)-*good curves*. Those are the closed oriented geodesics whose complex translation length $\mathbf{l}(\gamma)$ is 2ϵ-close to $2R$.

Let P_R be the planar oriented hyperbolic pair of pants whose cuffs C_i have length exactly $2R$ for $i \in \mathbf{Z}/3$. We define the space $\mathbf{\Pi}_{\epsilon,R}$ of (ϵ, R)-good pants to be the space of equivalence classes of maps $f : P_R \to M$ so that $f(C_i)$ is homotopic to an element of $\mathbf{\Gamma}_{\epsilon,R}$,

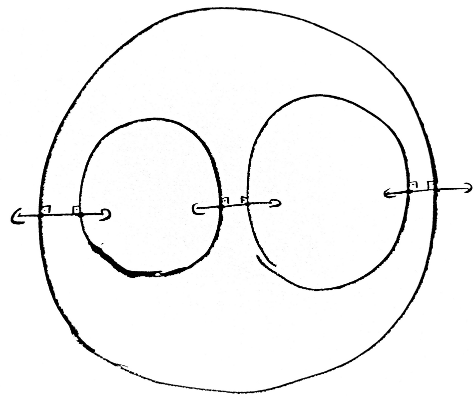

Figure 3.1: Short orthogeodesics and feet of a good pants.

for all $i \in \mathbf{Z}/3$. Two representatives f and g of elements of $\mathbf{\Pi}_{\epsilon,R}$ are equivalent if f is homotopic to $g \circ \psi$ for some orientation-preserving homeomorphism $\psi : P_R \to P_R$.

We let $\widetilde{\mathbf{\Pi}}_{\epsilon,R}$ be the space of *ends of* (ϵ, R) *good pants*, which can be thought as good pants with a marked cuff. Precisely $\widetilde{\mathbf{\Pi}}_{\epsilon,R}$ is the space of equivalence classes of pairs $[(f, C_i)]$, where $f \in \mathbf{\Pi}_{\epsilon,R}$ and $C_i \subset \partial P_R$ is a cuff. We say two representatives (f, C_i) and (g, C_j) of elements of $\widetilde{\mathbf{\Pi}}_{\epsilon,R}$ are equivalent if f is homotopic to $g \circ \psi$, where $\psi : P_R \to P_R$ is an orientation-preserving homeomorphism $\psi : P_R \to P_R$ so that $\psi(C_i) = C_j$. Note that forgetting the cuff of $[(f, C_i)] \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}$ defines a three-to-one surjection from $e : \widetilde{\mathbf{\Pi}}_{\epsilon,R} \to \mathbf{\Pi}_{\epsilon,R}$. For $\pi \in \mathbf{\Pi}_{\epsilon,R}$, we call $e^{-1}(\pi)$ the *ends* of π.

For $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$, we let $\widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)$ be the $[(f, C_i)] \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}$ so that $f(C_i)$ is homotopic to γ or its orientation reversal γ^{-1}. We can decompose $\widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)$ into $\mathbf{\Pi}^+_{\epsilon,R}(\gamma) \sqcup \mathbf{\Pi}^-_{\epsilon,R}(\gamma)$, where $\mathbf{\Pi}^+_{\epsilon,R}(\gamma)$ consists of the $[(f, C_i)]$ with $f(C_i) \sim \gamma$ and $\mathbf{\Pi}^-_{\epsilon,R}(\gamma)$ consists of the (f, C_i) with $f(C_i) \sim \gamma^{-1}$. There is a bijection

$$r : \mathbf{\Pi}^-_{\epsilon,R}(\gamma) \longrightarrow \mathbf{\Pi}^+_{\epsilon,R}(\gamma)$$

given by $r([(f, C_i)]) = ([(f \circ \rho, C_i)])$, where $\rho : P_R \to P_R$ is the reflection along the short orthogeodesics of P_R. We let $\mathbf{\Pi}_{\epsilon,R}(\gamma)$ denote the quotient of $\widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)$ by r.

The planar pair of pants P_R is equipped with six oriented *short orthogeodesics* a_{ij}, which are the geodesic segments connecting the cuffs C_i

The planar pair of pants P_R is equipped with three *short orthogeodesics*, which are the orthogonal geodesic segments from one cuff to another. The short orthogeodesic from C_i to C_j is denoted a_{ij}. A marked pair of pants $\pi \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}$ comes with left and right short orthogeodesics, respectively denoted $\eta^{\ell}(\pi)$ and $\eta^{r}(\pi)$, which are defined as follows. Choose a representative $(f, C_i) \in \pi$ that sends cuffs $C_j \subset \partial P_R$ to geodesics $\gamma_j \subset M$. We let $\eta^{\ell}(\pi)$ be the geodesic segment homotopic to $f(a_{i,i+1})$ (through segments from γ_i to γ_{i-1}) meeting γ_i and γ_{i-1} orthogonally. Similarly, $\eta^{r}(\pi)$ is the geodesic segment homotopic to $f(a_{i,i-1})$ meeting γ_i and γ_{i+1} orthogonally. Note that these definitions do not depend on the choice of representative in π.

We endow the short orthogeodesics of π with unit speed parametrizations, and from their construction, they are oriented to go from γ_i to the other cuffs. The *feet* of a short orthogeodesic γ of π are the unit vectors $-\eta'(0)$ and $\eta'(\ell(\eta))$. We call $\mathbf{ft}^{\ell}(\pi) = -(\eta^{\ell}(\pi))'(0)$ and $\mathbf{ft}^{r}(\pi) = -(\eta^{r}(\pi))'(0)$ respectively the left and right foot of π.

We define the *half length* $\mathbf{hl}(\gamma_i)$ of γ_i to be the complex distance between lifts of $\eta_{i,i-1}$ and $\eta_{i,i+1}$ to $\mathbf{H}^3$ that differ by a positively oriented segment of γ joining $\eta_{i,i-1}$ to $\eta_{i,i+1}$. It turns out that $\mathbf{l}(\gamma) = 2\mathbf{hl}(\gamma)$ (see [10], Section 2.8).

The unit normal bundle $\mathrm{N}^1(\gamma)$ to a oriented closed geodesic γ in M is acted upon simply and transitively by the group $\mathbf{C}/(\mathbf{l}(\gamma) + 2\pi i\mathbf{Z})$. We define $\mathrm{N}^1(\sqrt{\gamma})$ to be the quotient of $\mathrm{N}^1(\gamma)$ by the involution $n \mapsto n + \mathbf{hl}(\gamma)$. This is acted upon simply and transitively by $\mathbf{C}/(\mathbf{hl}(\gamma) + 2\pi i\mathbf{Z})$.

As $\mathbf{hl}(\gamma) = \mathbf{l}(\gamma)/2$, the left and right feet of $\pi \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)$ turn out to define the same point

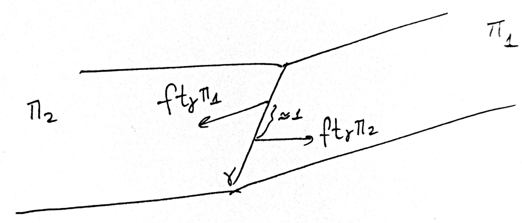

Figure 3.2: A good gluing between π_1 and π_2.

in $\mathrm{N}^1(\sqrt{\gamma})$. We thus have a map

$$\mathbf{ft} : \widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma) \longrightarrow \mathrm{N}^1\left(\sqrt{\gamma}\right)$$

which assigns the pants in π to its foot in $\mathrm{N}^1(\sqrt{\gamma})$. This map is also well defined on the unoriented version $\mathbf{\Pi}_{\epsilon,R}(\gamma)$.

Two pants $\pi_1, \pi_2 \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)$ that induce opposite orientations on γ are (ϵ, R)-*well matched* or *well glued* along $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$ if

$$\mathrm{dist}_{\mathrm{N}^1(\sqrt{\gamma})}(\mathbf{ft}\,\pi_1, \tau(\mathbf{ft}\,\pi_2)) < \frac{\epsilon}{R},$$

where τ is the translation of $\mathrm{N}^1(\sqrt{\gamma})$ given by $\tau(x) = x + 1 + i\pi$. In other words, the shearing between the feet is always approximately one (and the $i\pi$ takes into account that they point toward nearly opposite directions). Heuristically, the nearly constant shearing ensures you are never gluing the thin part of a pants (near the short orthogeodesics) to the thin part of another pants repeatedly.

For a finite set X, we let $\mathscr{M}(X)$ be the space of measures on X that are valued on the nonnegative integers. For $\mu \in \mathscr{M}(X)$, we let $\mathscr{S}(\mu)$ be the multiset consisting of $\mu(x)$ copies of each $x \in X$. For $\mu \in \mathscr{M}(\mathbf{\Pi}_{\epsilon,R})$, we let $\tilde{\mu} \in \mathscr{M}(\widetilde{\mathbf{\Pi}}_{\epsilon,R})$ denote the measure so that $\tilde{\mu}(\tilde{\pi}) = \mu(\pi)$ for any end $\tilde{\pi}$ of π. For $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$ and $\mathscr{F}$ a multiset of elements of $\widetilde{\mathbf{\Pi}}_{\epsilon,R}$, we let $\mathscr{F}_\gamma$ consists of the elements of $\mathscr{F}$ that also lie in $\widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)$. The multiset $\mathscr{S}_\gamma(\tilde{\mu})$ decomposes into a disjoint union $\mathscr{F}_\gamma^- \sqcup \mathscr{F}_\gamma^+$ of the ends reversing and preserving the orientation of μ. There is a map

$$\partial : \mathscr{M}(\mathbf{\Pi}_{\epsilon,R}) \longrightarrow \mathscr{M}(\mathbf{\Gamma}_{\epsilon,R})$$

defined via $\partial\mu(\gamma) = \sum_{\pi \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)} \mu(\pi)$.

We say a surface is *built out of* μ if it is obtained from gluing the elements of a submultiset of ends $\mathscr{F} \subset \mathscr{S}(\tilde{\mu})$ via bijections $\sigma_\gamma : \mathscr{F}_\gamma^- \to \mathscr{F}_\gamma^+$ for every cuff $\gamma \in \operatorname{supp} \partial\mu$. A surface is (ϵ, R)-*well built* out of μ if all the gluings are (ϵ, R)-good.

3.2 Assembling the surface

The first step in the construction is to show that a surface made out of good pants glued via good gluings is essential and nearly closed. Precisely,

Theorem 3.2.1. *Let $\mu \in \mathscr{M}(\mathbf{\Pi}_{\epsilon,R})$ be so that a closed surface S may be (ϵ, R)-well built from μ. Then, S is essential and $(1 + O(\epsilon))$-quasifuchsian.*

Proof. The proof of this is long and is the content of Section 2 of [8]. A more concise proof that ρ is $K(\epsilon)$-quasifuchsian, with $K(\epsilon) \to 1$ as $\epsilon \to 0$ (without the quantitative statement that $K(\epsilon) = 1 + O(\epsilon)$) can be found in the appendix of [10]. □

It remains to find such a measure $\mu \in \mathscr{M}(\mathbf{\Pi}_{\epsilon,R})$ from which we can build a closed surface with good gluings. The matching theorem below tells us we can take μ to be the measure $\mu_{\epsilon,R}$ that gives weight 1 to each $\pi \in \mathbf{\Pi}_{\epsilon,R}$.

Theorem 3.2.2. *For $\epsilon > 0$ sufficiently small, there is $R \geq R_0(\epsilon)$ so if $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$, there is a bijection*

$$\sigma_\gamma : \mathbf{\Pi}^-_{\epsilon,R}(\gamma) \longrightarrow \mathbf{\Pi}^+_{\epsilon,R}(\gamma)$$

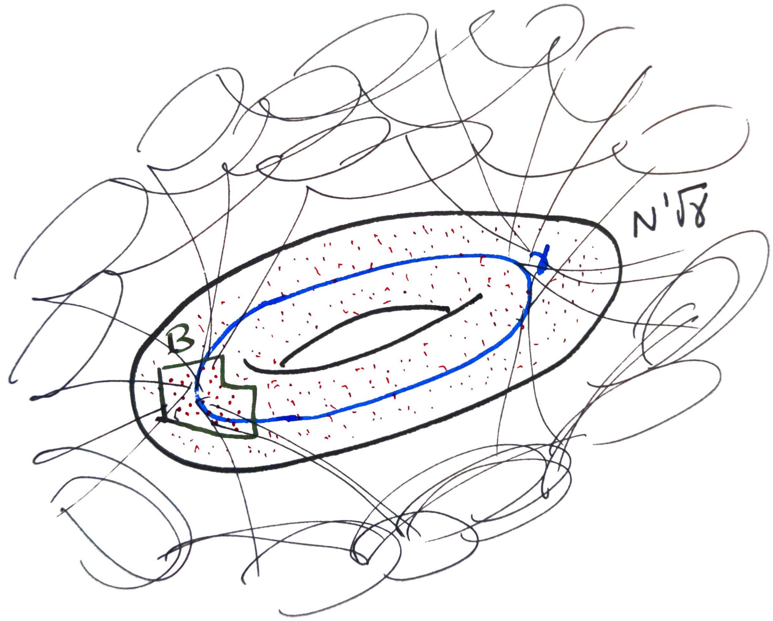

Figure 3.3: The feet of the good pants with cuff γ are well distributed in $\mathrm{N}^1(\sqrt{\gamma})$.

with the property that π is (ϵ, R)-well matched to $\sigma_\gamma(\pi)$ for all $\pi \in \mathbf{\Pi}^-_{\epsilon,R}(\gamma)$.

Gluing the pants in $\widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)$ via σ_γ for every γ gives us a closed surface, which by Theorem 3.2.1 is essential and $(1 + O(\epsilon))$-quasifuchsian.

The crucial ingredient in the proof of the matching theorem is the fact that the feet of pants in $\mathbf{\Pi}_{\epsilon,R}$ are well distributed in $\mathrm{N}^1(\sqrt{\gamma})$ for every $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$. This is the content of the equidistribution theorem below.

Theorem 3.2.3 (Equidistribution of feet)**.** *There is $q = q(M) > 0$ so that if $\epsilon > 0$ is small enough and $R > R_0(\epsilon)$, the following holds. Let $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$. If $B \subset \mathrm{N}^1(\sqrt{\gamma})$, then*

$$(1 - \delta)\lambda(N_{-\delta}B) \leq \frac{\#\{\pi \in \mathbf{\Pi}_{\epsilon,R}(\gamma) : \mathbf{ft}\,\pi \in B\}}{C(\epsilon, R, \gamma)} \leq (1 + \delta)\lambda(N_\delta B),$$

where $\lambda = \lambda_\gamma$ is the probability Lebesgue measure on $\mathrm{N}^1(\sqrt{\gamma})$, $\delta = e^{-qR}$, $N_\delta(B)$ is the δ-

neighborhood of B*,* $N_{-\delta}(B)$ *is the complement of* $N_\delta(\mathrm{N}^1(\sqrt{\gamma}) - B)$ *and* $C(\epsilon, R, \gamma)$ *is a constant depending only on* ϵ*,* R *and* $\mathbf{l}(\gamma)$*.*

Proof. The proof of the equidistribution of feet is the content of [11]. The main engine is the mixing of the frame flow in $\mathrm{Fr}\, M$. We use need a slight generalization of this theorem in Chapters 5 and 6, which is explained then. □

To complete the exposition, we will include a proof of the matching theorem using the equidistribution of feet along a curve. This is a relatively short argument which uses the Hall marriage theorem of combinatorics. Before stating it, we fix some notation: in a graph X, we wrtie $v \sim w$ for two vertices v and w that are connected by an edge. For a set A of vertices, we let $\partial N_1(A)$ be the vertices $w \notin A$ satisfying $w \sim v$ for some $v \in A$.

Theorem 3.2.4 (Hall marriage)**.** *Suppose* X *is a bipartite graph, i.e., the vertices* V *of* X *are the disjoint union of* V_1 *and* V_2*, where no two elements of a given* V_i *are connected by an edge. Then, there is a* matching $m : V_1 \to V_2$*, namely an injection so that* $v \sim m(v)$ *if and only if*

$$\#A \leq \#\partial N_1(A)$$

for any finite $A \subset V_1$*.*

We will also use the following fact

Proposition 3.2.5. *Let* M *be a Riemannian manifold equipped with a volume measure* $|\cdot|$ *and suppose* $A \subset M$ *satisfies* $|N_\eta(A)| \leq 1/2$ *for some* $\eta > 0$*. Then,*

$$\frac{|N_\eta(A)|}{|A|} \geq 1 + \eta h(M),$$

where $h(M)$ *is the Cheeger constant of* M*.*

Proof.

$$|N_\eta A - A| = \int_0^\eta |\partial N_t A|\, dt \geq \eta\, h(M)\, |A|.$$

□

It can be shown that the Cheeger constant of the flat torus $\mathrm{N}^1(\sqrt{\gamma})$ satisfies $h\left(\mathrm{N}^1(\sqrt{\gamma})\right) > 1/R$. (See, for example, [6].) Thus, if $A \subset \mathrm{N}^1(\sqrt{\gamma})$ satisfies $\lambda(N_\eta(A)) \leq 1/2$, we have

$$\frac{\lambda(N_\eta(A))}{\lambda(A)} > 1 + \frac{\eta}{R}. \tag{3.2.6}$$

We also define

$$\mathrm{Ft}\, B := \#\{\pi \in \mathbf{\Pi}_{\epsilon,R}(\gamma) : \mathbf{ft}\, \pi \in B\}.$$

The inequality 3.2.6, together with the equidistribution of feet gives us

Lemma 3.2.7. *Let $B \subset \mathrm{N}^1(\sqrt{\gamma})$ and let $\rho : \mathrm{N}^1(\sqrt{\gamma}) \to \mathrm{N}^1(\sqrt{\gamma})$ be a translation. Then,*

$$\mathrm{Ft}\, B \leq \mathrm{Ft}\, \rho\left(N_{\epsilon/R} B\right).$$

Proof. From the equidistribution of feet and the fact that ρ is measure preserving, we have that

$$(1-\delta)\lambda\left(N_{\epsilon/R-\delta} B\right) \leq \frac{\mathrm{Ft}\, \rho(N_{\epsilon/R} B)}{C_{\epsilon,R,\gamma}}.$$

Thus, it suffices to show that

$$\frac{\mathrm{Ft}\, B}{C_{\epsilon,R,\gamma}} \leq (1-\delta)\, \lambda\left(N_{\epsilon/R-\delta} B\right). \tag{3.2.8}$$

Using the equidistribution of feet again, we have that

$$\frac{\mathrm{Ft}\, B}{C_{\epsilon,R,\gamma}} \leq (1+\delta)\lambda\left(N_\delta B\right).$$

This reduces our goal to showing

$$\lambda\left(N_\delta B\right) \leq \frac{1-\delta}{1+\delta}\,\lambda\left(N_{\epsilon/R-\delta} B\right). \tag{3.2.9}$$

Suppose now that $\lambda(N_{\epsilon/2R}B) \leq 1/2$. From equation 3.2.6, we have that

$$\lambda\left(N_\delta B\right) < \frac{1}{1+\left(\frac{\epsilon}{2R}-\delta\right)\frac{1}{R}}\lambda\left(N_{\epsilon/2R} B\right).$$

But if R is large enough, as $\delta = e^{-qR}$, we have[1]

$$\frac{1}{1+\left(\frac{\epsilon}{2R}-\delta\right)\frac{1}{R}} \leq \frac{1-\delta}{1+\delta}.$$

Thus, we conclude that 3.2.9 holds if $\lambda(N_{\epsilon/R}B) \leq 1/2$. In particular,

$$\mathrm{Ft}\, B \leq \mathrm{Ft}\, \rho(N_{\epsilon/R}B)$$

holds in this case.

On the other hand, if $\lambda(N_{\epsilon/2R}B) > 1/2$, let $C = \mathrm{N}^1(\sqrt{\gamma}) - N_{\epsilon/R}\rho(B)$. Then, $\lambda(N_{\epsilon/2R}C) \leq 1/2$ and so by the same argument as above, for C instead of B and ρ^{-1} instead of ρ, we have

$$\mathrm{Ft}\, C \leq \mathrm{Ft}\, \rho^{-1}(N_{\epsilon/R}C).$$

---

[1]

$$\frac{1}{1+\left(\frac{\epsilon}{2R}-\delta\right)\frac{1}{R}} \leq \frac{1}{1+\frac{\epsilon}{3R^2}} \leq 1-\frac{\epsilon}{6R^2} \leq 1-3\delta \leq \frac{1-\delta}{1+\delta}.$$

But $\operatorname{Ft} C = \operatorname{Ft} \mathrm{N}^1(\sqrt{\gamma}) - \operatorname{Ft} \rho(N_{\epsilon/R} B)$ and $\operatorname{Ft} \rho^{-1}(N_{\epsilon/R} C) = \operatorname{Ft} \mathrm{N}^1(\sqrt{\gamma}) - \operatorname{Ft} B$. Therefore,

$$\operatorname{Ft} B \leq \operatorname{Ft} \rho(N_{\epsilon/R} B),$$

in this case as well. This completes the proof of the lemma. $\square$

Proof of the matching theorem. For $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$, we can make $\widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)$ into a graph by saying that $\pi_1 \sim \pi_2$ if π_1 and π_2 are (ϵ, R)-well matched, namely if they induce opposite orientations on γ and $\operatorname{dist}_{\mathrm{N}^1(\sqrt{\gamma})}(\mathbf{ft}\, \pi_1, \tau(\mathbf{ft}\, \pi_2)) < \epsilon/R$, where $\tau : \mathrm{N}^1(\sqrt{\gamma}) \to \mathrm{N}^1(\sqrt{\gamma})$ is the translation $\tau(x) = x + 1 + i\pi$. Since only the pants inducing opposite orientations on γ may be matched, $\widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma) = \mathbf{\Pi}^-_{\epsilon,R}(\gamma) \sqcup \mathbf{\Pi}^+_{\epsilon,R}(\gamma)$ is a bipartite graph. We wish to show there is a matching

$$\sigma_\gamma : \mathbf{\Pi}^-_{\epsilon,R}(\gamma) \longrightarrow \mathbf{\Pi}^+_{\epsilon,R}(\gamma).$$

By the Hall marriage theorem, it suffices to show that for $A \subset \mathbf{\Pi}^-_{\epsilon,R}(\gamma)$,

$$\begin{aligned}
\#A &\leq \partial N_1(A) \\
&= \#\{\pi^+ \in \mathbf{\Pi}^+_{\epsilon,R}(\gamma) : |\,\mathbf{ft}\, \pi^+ - \tau(\mathbf{ft}\, \pi^-)| < \epsilon/R \text{ for some } \pi^- \in A\} \\
&= \operatorname{Ft} \tau(N_{\epsilon/R}\, \mathbf{ft}\, A).
\end{aligned}$$

This, in turn, follows from Lemma 3.2.7 for $B = \mathbf{ft}\, A$ and $\rho = \tau$.

Since the sets $\mathbf{\Pi}^-_{\epsilon,R}(\gamma)$ and $\mathbf{\Pi}^+_{\epsilon,R}(\gamma)$ are finite and have the same cardinality, it follows that σ_γ is a bijection, which concludes the proof of the matching Theorem 3.2.2. $\square$

In summary, we have shown the matching Theorem 3.2.2, which allows us to build a closed $(1 + O(\epsilon))$-quasifuchsian surface $S_{\epsilon,R}$ in M by gluing one copy of each pants in $\mathbf{\Pi}_{\epsilon,R}$ via (ϵ, R)-good gluings.

Chapter 4

Connected surfaces going through every good pants

Recall that $\mu_{\epsilon,R} \in \mathscr{M}(\mathbf{\Pi}_{\epsilon,R})$ is the measure so that $\mu_{\epsilon,R}(\pi) = 1$ for each $\pi \in \mathbf{\Pi}_{\epsilon,R}$. In the previous chapter, we have seen that a closed, oriented, essential and $(1 + O(\epsilon))$-quasifuchsian surface $S_{\epsilon,R}$ may be built from $\mu_{\epsilon,R}$. We do not know, however, whether $S_{\epsilon,R}$ is connected, or what its components may look like. Following ideas of Liu and Markovic [14], we show that if we take $N = N(\epsilon, R, M)$ copies of $S_{\epsilon,R}$, it is possible to perform cut-and-paste surgeries around certain good curves in order to get *connected* closed, oriented, essential and $(1 + O(\epsilon))$-quasifuchsian surfaces $\hat{S}_{\epsilon,R}$.

Theorem 4.0.1. *There is an integer $N = N(\epsilon, R, M) > 0$ so that a* connected*, closed, oriented, essential and $(1 + O(\epsilon))$-quasifuchsian surface may be built from $N\mu_{\epsilon,R}$.*

We define a measure $\mu \in \mathscr{M}(\mathbf{\Pi}_{\epsilon,R})$ to be *irreducible* if for any nontrivial decomposition $\mu = \mu_1 + \mu_2$, there is a curve $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$ so that γ lies in $\operatorname{supp} \partial\mu_1$ and its orientation reversal γ^{-1} lies in $\operatorname{supp} \partial\mu_2$.

If μ is *not* irreducible, then no connected surface may be built from μ. In fact, if there is a nontrivial decomposition $\mu = \mu_1 + \mu_2$ so that if $\gamma \in \operatorname{supp} \partial\mu_1$, then $\gamma^{-1} \notin \operatorname{supp} \partial\mu_2$, then no pants in $\operatorname{supp} \partial\mu_1$ may be glued to pants in $\operatorname{supp} \partial\mu_2$. Thus, a surface built out of μ will have at least two components.

On the other hand, if μ is irreducible, we have the following theorem, which is close to Lemma 3.9 of Liu and Markovic [14]. (They do not assume μ to be positive on all pants, using a weaker hypothesis instead, but the conclusion is the same.)

Theorem 4.0.2. *Suppose $\mu \in \mathscr{M}(\mathbf{\Pi}_{\epsilon,R})$ is an irreducible measure so that $\mu(\pi) > 0$ for every $\pi \in \mathbf{\Pi}_{\epsilon,R}$ and a closed surface may be (ϵ, R)-well built from μ. Then, there is an integer $N = N(\mu)$ so that a* connected *closed surface may be $(2\epsilon, R)$-well built from $N\mu$.*

In view of that, our goal for this chapter is to prove that $\mu_{\epsilon,R}$ is irreducible. Fortunately, we have the following theorem, which is Proposition 7.1 of [14].

Proposition 4.0.3. *Given two curves $\gamma_0, \gamma_1 \in \mathbf{\Gamma}_{\epsilon,R}$, we may find a sequence of pants $\pi_0, \ldots, \pi_n$ in $\mathbf{\Pi}_{\epsilon,R}$ so that γ_0 is a cuff of π_0, γ_1 is a cuff of π_n and π_i may be glued to π_{i+1} for $0 \le i < n$.*

The gluings in the proposition above are not necessarily (ϵ, R)-good, as suggested in Figure 4.1.

Proof sketch. Liu and Markovic argue that γ_0 and γ_1 are, respectively, the boundaries of pants π_0 and π_1 that have cuffs in $\mathbf{\Gamma}_{\epsilon/10000,R}$. For $i \in \{0, 1\}$, the pants π_i is obtained by taking an appropriate self-orthogeodesic segment σ_i of γ_i and homotoping the closed curves comprising of the pieces of γ_i between the endpoints of σ_i and σ_i to good cuffs. Thus, this reduces the problem to showing that any two curves $\gamma_0, \gamma_1 \in \mathbf{\Gamma}_{\epsilon/10000,R}$ may be connected via pants in $\mathbf{\Pi}_{\epsilon,R}$. This is done by an yet trickier geometric construction, called *swapping*. Roughly, using mixing of the frame flow, the curves γ_0 and γ_1 are joined by two segments of length approximately $R/2$. These segments are chosen in a way that they, together with

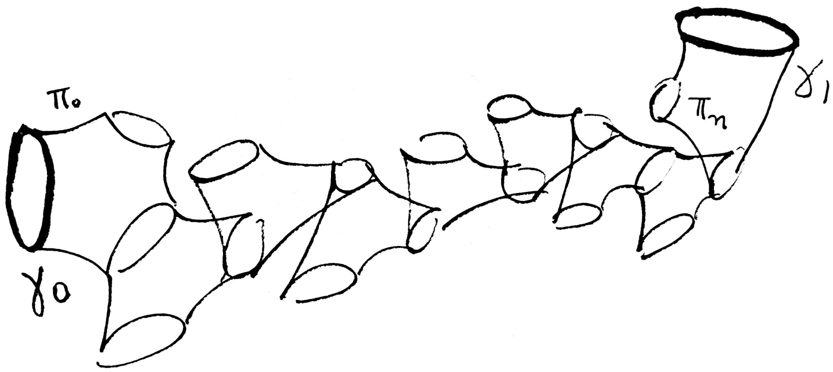

Figure 4.1: Proposition 4.0.3 tells us we may connect two good curves γ_0 and γ_1 via a bridge of good pants $\pi_0, \ldots, \pi_n$. As suggested in the picture, the gluings are not necessarily (ϵ, R)-good.

γ_0 and γ_1 lie on a surface F, that has bounded genus and exactly 4 boundary components. This surface F, in turn, admits a pants decomposition by pants in $\mathbf{\Pi}_{\epsilon,R}$. $\square$

Theorem. *The measure $\mu_{\epsilon,R}$ is irreducible.*

Proof. Let $\mu_{\epsilon,R} = \mu_0 + \mu_1$ be a nontrivial decomposition. Let $\gamma_0 \in \operatorname{supp} \partial\mu_0$ and $\gamma_1 \in \operatorname{supp} \partial\mu_1$. In view of Proposition 4.0.3, there are pants $\pi_0, \ldots, \pi_n$ in $\mathbf{\Pi}_{\epsilon,R} = \operatorname{supp} \mu_{\epsilon,R}$ so that γ_0 is a cuff of π_0, γ_1 is a cuff of π_n and π_i may be glued to π_{i+1}. This means there is a curve γ, which is a cuff of some π_i, so that $\gamma \in \operatorname{supp} \mu_1$ and $\gamma^{-1} \in \operatorname{supp} \mu_2$. This means $\mu_{\epsilon,R}$ is irreducible. $\square$

We conclude the chapter by providing a proof of Theorem 4.0.2. The regluing of surfaces featured in this proof provides inspiration for the construction of non-equidistributing surfaces in Chapter 7.

We start with the following lemma about pants decompositions.

Lemma 4.0.4. *Let S be a surface with a pants decomposition P. Then, S has a double cover $\hat{S}$ to which the pants in P lift homeomorphically to pants with nonseparating cuffs.*

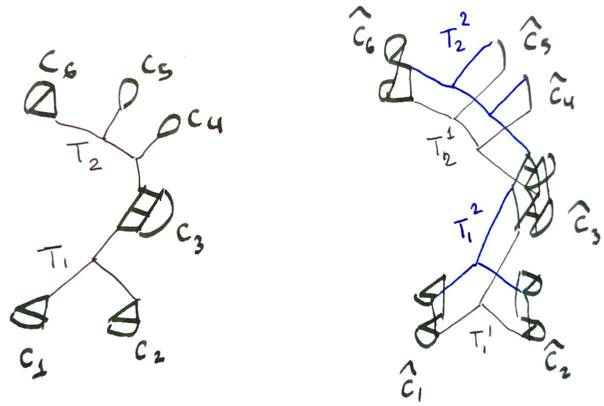

Figure 4.2: Proof of Lemma 4.0.4. On the left, we have the dual graph X to the pants decomposition P of S. On the right, we have the double cover $\hat{X}$. T_i^1 and T_i^2 are the lifts of T_i to $\hat{X}$, which are nonseparating in $\hat{X}$.

Proof. Let X be the graph whose the vertices are pants in P and the edges are the cuffs shared by pants in P. A cuff is separating in S if and only if its corresponding edge in X is separating. Thus, our task is to show that X has a double cover $\hat{X}$ that only has nonseparating edges.

To do so, let $F = \sqcup_{i=1}^{n} T_i \subset X$ be the graph-theoretic forest consisting of all separating edges of X, where the T_i are disjoint trees. We also write $X - F = \sqcup_{j=1}^{m} C_j$, where C_j are disjoint connected components. For each j, we take a double cover $d_j : \hat{C}_j \to C_j$. This gives us a double cover $d : \sqcup_{j=1}^{m} \hat{C}_j \to \sqcup_{j=1}^{m} C_j$.

Note that each $\hat{C}_j$ consists of nonseparating edge. If some $\hat{C}_j$ had a separating edge e, it would have another separating edge e', the image of e under the nontrivial deck transformation $\hat{C}_j \to \hat{C}_j$. Thus, $\hat{C}_j - \{e, e'\}$ consists of *three* components, otherwise one of e or e' would not be separating. In particular, the inverse image of the (connected) set $C_j - d_j(e)$ under d_j would consist of three compoents, contradicting the fact that d_j is a double cover.

For each i, we attach a copy of T_i to each of the two lifts that ∂T_i has in $\sqcup_{i=1}^{m} \hat{C}_j$. As

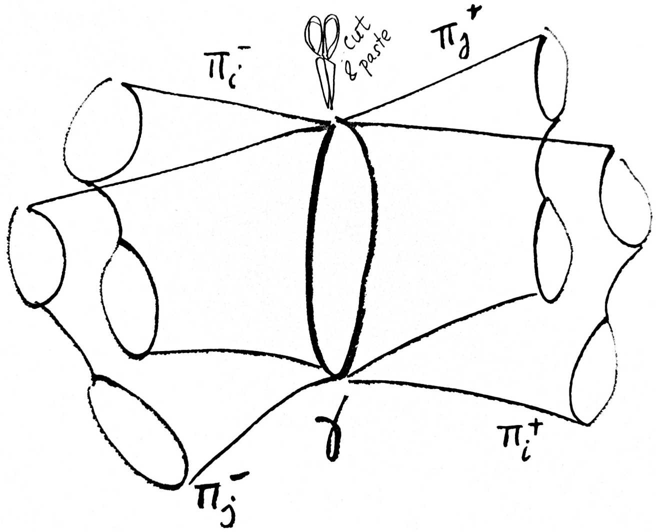

Figure 4.3: Regluing S_1 to S_2 with a good gluing and reducing the number of components of S.

a result, we get a double cover $D : \hat{X} \to X$ which extends d. (See Figure 4.2.) The trees $T_i \subset X$ have nonseparating lifts to $\hat{X}$, as both of their lifts are bounded by the same subset of the $\{\hat{C}_j\}_{j=1}^m$. □

Proof of Theorem 4.0.2. Suppose the closed oriented essential $(1 + O(\epsilon))$-quasifuchsian surface S we build out of $\mu_{\epsilon,R}$ has r components:

$$S = S_1 \sqcup \cdots \sqcup S_r.$$

We take a finite covering $\hat{S} \to S$ of degree $N = N(\epsilon, R)$ so that the good pants lift homeomorphically and each good curve appears at least r times in each component. In view of Lemma 4.2, we may assume that the good curves lift to nonseparating curves in $\hat{S}$.

Due to the irreducibility of $\mu_{\epsilon,R}$, there is a cuff $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$ that is shared by at least two components. Let $\Pi_k(\gamma)$ be the ends of pants in $\widetilde{\mathbf{\Pi}}_{\epsilon,R}(\gamma)$ that lie in S_k. We can divide

$\Pi_k(\gamma) = \Pi_k^-(\gamma) \sqcup \Pi_k^+(\gamma)$ that induce a negative and positive orientation on γ. We wish to show

Claim. *There is $i \neq j$ so that there are pants $\pi_i^- \in \Pi_i^-(\gamma)$ and $\pi_j^+ \in \Pi_j^+(\gamma)$ so that*

$$|\,\mathbf{ft}_\gamma\, \pi_i^- - \mathbf{ft}_\gamma\, \pi_j^+| < \frac{\epsilon}{R}.$$

From the claim, since π_k^- is (ϵ, R)-well glued to some $\pi_k^+ \in \Pi_k^+(\gamma)$ for $k \in \{i, j\}$, we have that

$$|\,\mathbf{ft}\,\pi_i^- - \tau(\mathbf{ft}\,\pi_j^+)| < \frac{2\epsilon}{R} \quad \text{and} \quad |\,\mathbf{ft}\,\pi_i^+ - \tau(\mathbf{ft}\,\pi_j^-)| < \frac{2\epsilon}{R}$$

and so $\pi_i^\mp$ may be $(2\epsilon, R)$-well glued to $\pi_j^\pm$, where $\tau(x) = x + 1 + i\pi$. Performing this regluing reduces the number of components of $\hat{S}$, and continuing this process we obtain a connected surface with the desired properties.

We conclude by proving the claim. Let $U_k := N_{\epsilon/2R}(\mathbf{ft}\,\Pi_k^-(\gamma))$. From the equidistribution of feet, we have that $\bigcup_{k=1}^r U_k = \mathrm{N}^1(\sqrt{\gamma})$. Indeed, if we let $F = \mathbf{ft}(\mathbf{\Pi}^-_{\epsilon,R}(\gamma))$, we have that

$$0 = \frac{\mathrm{Ft}(\mathrm{N}^1(\sqrt{\gamma}) - F)}{\#\mathbf{\Pi}_{\epsilon,R}(\gamma)} \geq (1-\delta)|N_{-\delta}(\mathrm{N}^1(\sqrt{\gamma}) - F)|,$$

which implies that $N_\delta(F)$ has full measure. Thus, as $\epsilon/2R > \delta = e^{-qR}$, we conclude that $N_{\epsilon/R}(F) = \bigcup_{k=1}^r U_k$ is all of $\mathrm{N}^1(\sqrt{\gamma})$. But as $\mathrm{N}^1(\sqrt{\gamma})$ is connected and the U_k are open, there has to be an $i \neq j$ so that $U_i \cap U_j \neq \emptyset$. $\square$

Chapter 5

Barycenters of the good pants are equidistributed

Let $T \subset \mathbf{H}^3$ be an oriented ideal triangle. There are three horocycles based on the vertices of T that are pairwise tangent, with their tangency points lying in ∂T. The points where the horocycles meet ∂T are called the *midpoints* of the edges of T.

The geodesic rays from the midpoints of T towards the opposite vertices meet at the *barycenter* $b(T)$ of T. The *framed barycenters* of T are the frames (v, w, n) based at $b(T)$, where v points away from a side of T, n is normal to T and $v \times w = n$.

The barycenter of an ideal triangle $T \subset M$ is the projection onto M of the barycenter of a lift of T to $\mathbf{H}^3$. The framed barycenters of $T \subset M$ are the projections to $\operatorname{Fr} M$ of the framed barycenters of a lift of T to $\mathbf{H}^3$.

A good pants $\pi \in \mathbf{\Pi}_{\epsilon,R}$ has a pleated structure consisting of two ideal triangles, as in Figure 5.2. Its *barycenters* are the framed barycenters of these ideal triangles.

We let $\beta_{\epsilon,R}$ be the weighted uniform probability measure supported on the barycenters of the pants in $\mathbf{\Pi}_{\epsilon,R}$. In this chapter, we will show

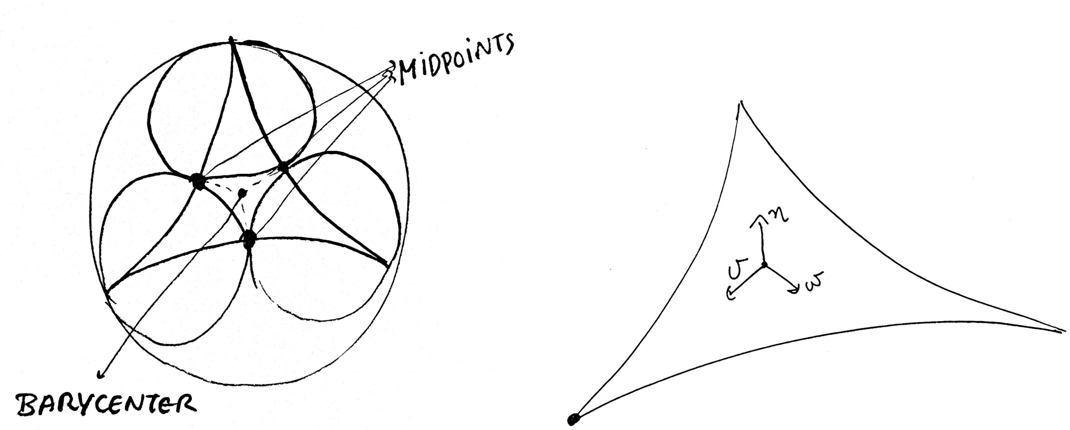

Figure 5.1: Left: midpoints and barycenter of an ideal triangle. Right: one of the three framed barycenters of an ideal triangle.

Theorem 5.0.1 (Equidistribution of barycenters)**.** *For $\epsilon \to 0$ and $R(\epsilon) \to \infty$ fast enough,*

$$\beta_{\epsilon,R(\epsilon)} \xrightarrow{\star} \nu_{\mathrm{Fr}\,M},$$

where $\nu_{\mathrm{Fr}\,M}$ is the probability volume measure on $\mathrm{Fr}\,M$*.*

In other words, the barycenters of the good pants equidistribute in $\mathrm{Fr}\,M$ as $\epsilon \to 0$ and $R(\epsilon) \to \infty$. This will be used in Chapter 6 to show that the connected surface $S_{\epsilon,R}$ built out of $N = N(\epsilon, R, M)$ copies of each pants in $\mathbf{\Pi}_{\epsilon,R}$ equidistributes as $\epsilon \to 0$ and $R \to \infty$. This will follow from the fact that the unit tangent bundle of each pair of pants (outside of the pleats) may be obtained from the barycenters via the right action of a set $\Delta \subset \mathrm{PSL}_2\,\mathbf{R}$.

5.1 Outline of the proof

We will show the equidistribution of barycenters, Theorem 5.1, in three steps.

First, we will prove that the feet of all pants in $\mathbf{\Pi}_{\epsilon,R}$, seen as points in $\mathrm{Fr}\,M$, equidistributes as $\epsilon \to 0$. Precisely, a foot f of $\pi = [(f, C_i)] \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}$ is associated to the frame

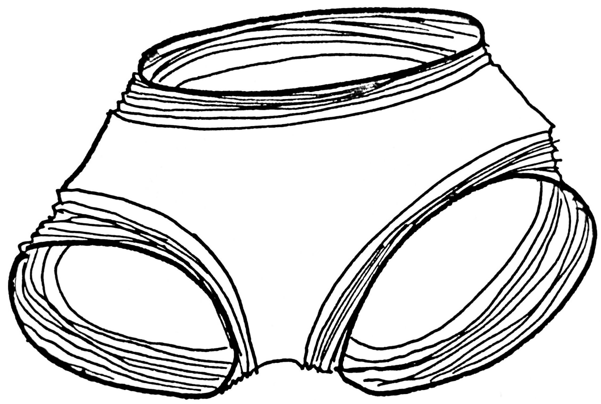

Figure 5.2: Pleated structure of a pair of pants consisting of two ideal triangles.

$(v, f, v \times f)$, where v is the unit tangent vector to the $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$ homotopic to $f(C_i)$. (With this identification, we can realize $\mathrm{N}^1(\gamma)$ as a subset of $\mathrm{Fr}\, M$.) We let $\phi_{\epsilon,R}$ be the weighted uniform probability measure on $\mathrm{Fr}\, M$ supported on the feet of pants in $\mathbf{\Pi}_{\epsilon,R}$. We will show

Lemma 5.1.1 (Equidistribution of feet in $\mathrm{Fr}\, M$)**.** *For $\epsilon \to 0$ and $R(\epsilon) \to \infty$ fast enough,*

$$\phi_{\epsilon,R(\epsilon)} \xrightarrow{\star} \nu_{\mathrm{Fr}\, M}.$$

The proof of Lemma 5.2 will use the fact that the feet are well-distributed in the unit normal bundle of a given good curve (due to Kahn-Wright [11], in a modified version), as well as the fact that the good curves themselves are asymptotically almost surely well-distributed in $\mathrm{T}^1\, M$ (due to Lalley [13]).

Let $a_t = \mathrm{diag}(e^{t/2}, e^{-t/2})$ and $k \in \mathrm{SO}_2$ be the ninety-degree rotation bringing the first vector in a frame to the second, fixing the third. The second step of the proof is to observe that the right action[1] $v_R := R_{a_{R/2}ka_{\log(\sqrt{3}/2)}}$ of the element

$$a_{R/2}\, k\, a_{\log(\sqrt{3}/2)} \in \mathrm{PSL}_2\,\mathbf{C},$$

brings the feet of a pants π to frames that are very close to the framed barycenters of the triangles of the pleated structure of π.

We call the images of the feet of $\mathbf{\Pi}_{\epsilon,R}$ under v_R the *approximate barycenters* of the pants in $\mathbf{\Pi}_{\epsilon,R}$. In Lemma 5.4.1, we show that the distances in $\mathrm{Fr}\,M$ between the approximate barycenters and the actual barycenters of pants in $\mathbf{\Pi}_{\epsilon,R}$ go to zero uniformly as $\epsilon \to 0$.

Let $\beta^a_{\epsilon,R}$ be the (weighted) uniform probability measure on the approximate barycenters of the pants in $\mathbf{\Pi}_{\epsilon,R}$. We will show that these approximate barycenters equidistribute in $\mathrm{Fr}\,M$, namely

Proposition 5.1.2. *[Equidistribution of approximate barycenters] For $\epsilon \to 0$ and $R(\epsilon) \to \infty$ fast enough,*

$$\beta^a_{\epsilon,R(\epsilon)} \xrightarrow{\star} \nu_{\mathrm{Fr}\,M}.$$

To conclude, we use Lemmas 5.4.1 and 5.1.2 to show the main theorem of the chapter – the *actual* barycenters of the pants equidistribute.

5.2 Left and right

In this section we will do some bookkeeping that will be useful to carry out the rest of the proof.

Let P_R be the oriented planar hyperbolic pair of pants whose cuffs have size $2R$, as defined in Chapter 3. The cuffs of P_R are named C_0, C_1 and C_2, as in Figure 5.3. As defined

[1] We choose an origin $o \in \mathrm{Fr}\,M$ and identify $\mathrm{Fr}\,M \cong \mathrm{PSL}_2\,\mathbf{C}$ by sending go to g. We say that the right action R_h of an element $h \in G$ on $go \in \mathrm{Fr}\,M$ is given by $R_h(go) = gho$. This is an antihomomorphism $R : G \to \mathrm{Aut}\,G$.

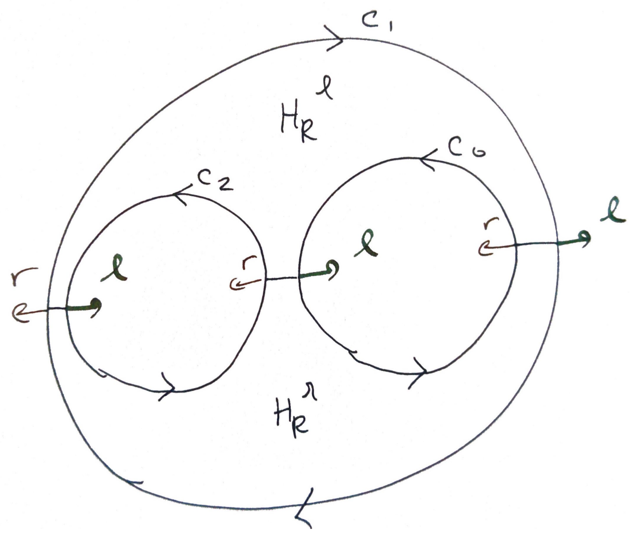

Figure 5.3: The pants P_R divided into left and right hexagons, with its left and right feet.

before, each cuff C_i has two *feet* in $\mathrm{N}^1(C_i)$, which are unit vectors in the direction of the short orthogeodesics incident to C_i. The *left* foot of C_i points towards C_{i-1} and the *right* foot points towards C_{i+1}.

We can cut P_R along its short orthogeodesics to obtain two right-angled hexagons H_R^ℓ and H_R^r. The *left* right-angled hexagon H_R^ℓ of P_R is the one so that a traveller going around ∂H_R^ℓ in the direction given by the orientation of P_R sees the cuffs in the cyclic order $(C_0\, C_1\, C_2)$. The *right* right-angled hexagon is the other one (associated to the cyclic order $(C_0\, C_2\, C_1)$).

As before, let v_R be the right action of $a_{R/2} k a_{\log(\sqrt{3}/2)} \in \mathrm{PSL}_2\,\mathbf{R}$. Observe that the image of a left foot of P_R under v_R falls inside H_R^ℓ. Similarly, the image of a right foot under v_R falls in H_R^r.

We can turn the right-angled hexagons H_R^ℓ and H_R^r into ideal triangles T_R^ℓ and T_R^r by spinning their vertices around the cuffs, following their orientation. See Figure 5.4.

Let $\pi \in \mathbf{\Pi}_{\epsilon,R}$ and $f \in \pi$ be a pleated representative (so $f(P_R)$ is made out of two

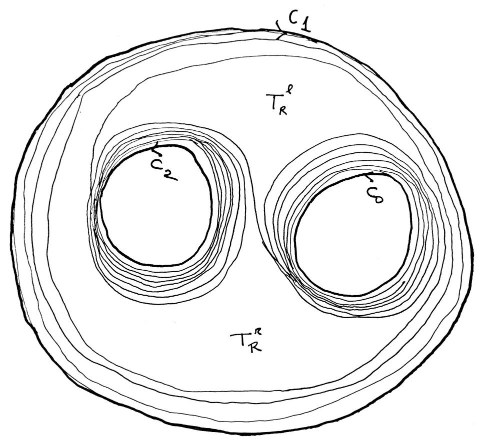

Figure 5.4: Spinning the hexagons of P_R into two ideal triangles. Picture drawn by Natalie Rose Schwartz.

ideal triangles). We call $f(T_R^\ell)$ the *left triangle* $T^\ell(\pi)$ of π and $f(T_R^r)$ the *right triangle* $T^r(\pi)$ of π. Note that these are well-defined as they do not depend on the choice of pleated representative in π.

Now let $\pi \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}$ and $(f, C_i) \in \pi$ be a pleated representative. The *left barycenter* of π, denoted $\mathbf{bar}^\ell(\pi)$, is the framed barycenter of $T^\ell(\pi)$ associated to the side $f(C_i)$. Similarly, the *right barycenter* of π, denoted $\mathbf{bar}^r(\pi)$, is the framed barycenter of $T^r(\pi)$ associated to the side $f(C_i)$.

5.3 Equidistribution of feet in $\mathrm{Fr}\, M$

The goal of this section is to prove Lemma 5.1.1, in other words, that

$$\phi_{\epsilon,R(\epsilon)} \xrightarrow{\star} \nu_{\mathrm{Fr}\, M}$$

as $\epsilon \to 0$. To do so, we will use the fact that the feet of pants are well-distributed along a good curve. This is Theorem 3.2.3, due to Kahn-Wright [10], but we will use the modified version of Theorem 5.3.1 below. The difference is that Theorem 3.2.3 is stated for counting feet in a subset B of $\mathrm{N}^1(\sqrt{\gamma})$, whereas the counting we will do is weighted by a nonnegative function $g \in L^\infty(\mathrm{N}^1(\gamma)) \subset L^\infty(\mathrm{Fr}\, M)$.

For $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$, we let λ^γ denote the probability Lebesgue measure in $\mathrm{N}^1(\gamma) \subset \mathrm{Fr}\, M$. For a bounded function g on a metric space, we let

$$m_\delta(g)(p) = \inf_{B_\delta(p)} g \quad \text{and} \quad M_\delta(g)(p) = \sup_{B_\delta(p)} g,$$

where $B_\delta(p)$ is the metric ball of radius δ around p.

Theorem 5.3.1. *[Equidistribution of feet along a curve] There exists $q > 0$ depending on M such that for any $\epsilon > 0$, there is $R \geq R_0(\epsilon)$ so that the following holds. Let $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$. If $g \in L^\infty\,(\mathrm{Fr}\, M)$ is a nonnegative function, then*

$$(1-\delta)\int_{\mathrm{N}^1(\gamma)} m_\delta(g)\, d\lambda^\gamma \leq \frac{1}{C_{\epsilon,R,\gamma}} \sum_{\pi \in \mathbf{\Pi}_{\epsilon,R}(\gamma)} \Big(g(\mathbf{ft}^\ell\, \pi) + g(\mathbf{ft}^r\, \pi) \Big) \leq (1+\delta)\int_{\mathrm{N}^1(\gamma)} M_\delta(g)\, d\lambda^\gamma, \quad (\star)$$

where $\delta = e^{-qR}$,

$$C_{\epsilon,R,\gamma} = \frac{2\pi c_\epsilon \epsilon^4 \ell(\gamma) e^{4R-\ell(\gamma)}}{\mathrm{vol}\, M}$$

and $c_\epsilon \overset{\epsilon\to 0}{\longrightarrow} 1$.

Proof from Theorem 3.2.3. Let $g \in L^\infty(\mathrm{Fr}\, M)$. Since the measure λ^γ is supported on $\mathrm{N}^1(\gamma)$, we may assume $g \in L^\infty(\mathrm{N}^1(\gamma))$. Let $h(n) := g(n) + g(n + \mathbf{hl}(\gamma))$. Since h is invariant under $n \mapsto n + \mathbf{hl}(\gamma)$, h descends to a function $\check{h} \in L^\infty(\mathrm{N}^1(\sqrt{\gamma}))$ so that $h \circ \mathrm{proj} = \check{h}$, where $\mathrm{proj} : \mathrm{N}^1(\gamma) \to \mathrm{N}^1(\sqrt{\gamma})$ is the quotient projection.

Using the shorthand notation

$$\{f > y\} := \{n \in \mathrm{N}^1(\sqrt{\gamma}) : f(n) > y\},$$

note that

$$N_{-\delta}\{f > y\} = \{m_\delta(f) > y\} \quad \text{and} \quad N_\delta\{f > y\} = \{M_\delta(f) > y\}.$$

Thus, Theorem 3.2.3 gives us

$$(1-\delta)\,\lambda(\{m_\delta(\check{h}) > y\}) \le \frac{\#\{\pi \in \mathbf{\Pi}_{\epsilon,R} : \check{h}(\mathbf{ft}\,\pi) > y\}}{C_{\epsilon,R,\gamma}} \le (1+\delta)\,\lambda(\{M_\delta(\check{h}) > y\}), \tag{5.3.2}$$

where λ is the probability Lebesgue measure on $\mathrm{N}^1(\sqrt{\gamma})$.

A basic property of the Lebesgue integral says that for a function $f \in L^\infty(X,\mu)$, where X is a space with a measure μ, we have $\int_X f\,d\mu = \int_0^{\|f\|_\infty} \mu(\{f > y\})\,dy$. Thus, if we integrate the inequality (5.3.2) above with respect to y from 0 to $\|\check{h}\|_{L^\infty(\mathrm{N}^1(\sqrt{\gamma}))}$ and apply this property for λ and the counting measure of feet in $\mathrm{N}^1(\sqrt{\gamma})$, we obtain

$$(1-\delta)\int_{\mathrm{N}^1(\sqrt{\gamma})} m_\delta(\check{h})\,d\lambda \le \frac{1}{C_{\epsilon,R,\gamma}} \sum_{\pi \in \mathbf{\Pi}_{\epsilon,R}(\gamma)} \check{h}(\mathbf{ft}\,\pi) \le (1+\delta)\int_{\mathrm{N}^1(\sqrt{\gamma})} M_\delta(\check{h})\,d\lambda.$$

Note that $\check{h}(\mathbf{ft}\,\pi) = g(\mathbf{ft}^\ell\,\pi) + g(\mathbf{ft}^r\,\pi)$, so the middle term of the inequality is the same as in ($\star$). On the other hand, $m_\delta(\check{h}) \circ \mathrm{proj} = m_\delta(h)$. Thus, $\int_{\mathrm{N}^1(\sqrt{\gamma})} m_\delta(\check{h})\,d\lambda = \int_{\mathrm{N}^1(\gamma)} m_\delta(h)\,d\lambda^\gamma$. Finally, $\int_{\mathrm{N}^1(\gamma)} m_\delta(h)\,d\lambda^\gamma \ge 2\int_{\mathrm{N}^1(\gamma)} m_\delta(g)\,d\lambda^\gamma$ and similarly $\int_{\mathrm{N}^1(\sqrt{\gamma})} M_\delta(\check{h})\,d\lambda \le 2\int_{\mathrm{N}^1(\gamma)} M_\delta(g)\,d\lambda^\gamma$. This yields the desired inequality ($\star$), up to the constant $C_{\epsilon,R,\gamma}$ absorbing a factor of 2. $\square$

We can simplify the main statement of the theorem with the following notations. For a measure μ on a space X, and $g \in L^\infty(X)$, we let $\mu(g) := \int_X g\,d\mu$. We define a measure $\phi^\gamma_{\epsilon,R}$ supported on $\mathrm{N}^1(\gamma)$ by

$$\phi^\gamma_{\epsilon,R}(g) = \frac{1}{C_{\epsilon,R,\gamma}} \sum_{\pi \in \mathbf{\Pi}_{\epsilon,R}(\gamma)} \left(g(\mathbf{ft}^\ell\,\pi) + g(\mathbf{ft}^r\,\pi)\right),$$

where $g \in C(\mathrm{Fr}\,M)$.

Fix a nonnegative function $g \in C(\operatorname{Fr} M)$. The inequality ($\star$) can be rewritten as

$$(1-\delta)\,\lambda^{\gamma}\,(m_{\delta} g) \leq \phi^{\gamma}_{\epsilon,R}(g) \leq (1+\delta)\,\lambda^{\gamma}\,(M_{\delta} g)$$

and we can average it over all $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$, yielding

$$(1-\delta)\frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \lambda^{\gamma}\,(m_{\delta} g) \leq \frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \phi^{\gamma}_{\epsilon,R}(g) \leq (1+\delta)\frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \lambda^{\gamma}\,(M_{\delta} g)\,.$$

If we can show that the upper and lower bounds of this inequality are very close to $\nu_{\operatorname{Fr} M}(g)$ and that the middle term is very close to $\phi_{\epsilon,R}(g)$ as $\epsilon \to 0$, then it will follow that $\phi_{\epsilon,R}(g) \overset{\epsilon\to 0}{\longrightarrow} \nu_{\operatorname{Fr} M}(g)$. This, in turn, implies Lemma 5.1.1, using the fact that since M is compact, $C(\operatorname{Fr} M) \subset L^{\infty}(\operatorname{Fr} M)$, as well as the fact that if $\phi_{\epsilon,R}(g) \overset{\epsilon\to 0}{\longrightarrow} \nu_{\operatorname{Fr} M}(g)$ for nonnegative functions $g \in C(\operatorname{Fr} M)$, it follows that $\phi_{\epsilon,R}(g) \overset{\epsilon\to 0}{\longrightarrow} \nu_{\operatorname{Fr} M}(g)$ for all $g \in C(\operatorname{Fr} M)$.

Our task is is therefore to show the following two lemmas:

Lemma 5.3.3. *For $g \in C(\operatorname{Fr} M)$,*

$$\left|\frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \lambda^{\gamma}\,(m_{\delta} g) - \nu_{\operatorname{Fr} M}(g)\right| \quad \textit{and} \quad \left|\frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \lambda^{\gamma}\,(M_{\delta} g) - \nu_{\operatorname{Fr} M}(g)\right|$$

converge to zero as $\epsilon \to 0$.

Lemma 5.3.4. *For $g \in C(\operatorname{Fr} M)$,*

$$\left|\frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \phi^{\gamma}_{\epsilon,R}(g) - \phi_{\epsilon,R}(g)\right| \overset{\epsilon\to 0}{\longrightarrow} 0.$$

Proof of Lemma 5.3.3. For $g \in C(\mathrm{Fr}\, M)$, we define a function $\hat{g} \in C(\mathrm{T}^1 M)$ via

$$\hat{g}(p,v) = \frac{1}{2\pi} \int_{S^1(v)} g(p,v,\theta)\, d\theta,$$

where $S^1(v)$ is the circle in $T^1_p M$ orthogonal to v. For $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$, we let $d\gamma$ be the probability length measure of γ on $\mathrm{T}^1(M)$, in other words, for $h \in C(\mathrm{T}^1 M)$.

$$\int_{\mathrm{T}^1 M} h\, d\gamma = \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} h(\gamma(t), \gamma'(t))\, dt.$$

Note that for $g \in C(\mathrm{Fr}\, M)$,

$$\int_{\mathrm{T}^1 M} \hat{g}\, d\gamma = \int_{\mathrm{Fr}\, M} g\, d\lambda^\gamma.$$

Let $\mathbf{Prob}_\epsilon$ be the uniform probability measure on $\mathbf{\Gamma}_{\epsilon,R}$. In Theorem II of [13], Lalley showed that, if $h \in C(\mathrm{T}^1 M)$ and $\eta > 0$, then

$$\mathbf{Prob}_\epsilon \left(\left| \int_{\mathrm{T}^1 M} h\, d\gamma - \nu_{\mathrm{T}^1 M}(h) \right| > \eta \right) \xrightarrow{\epsilon \to 0} 0.$$

In other words, if $g \in C(\mathrm{Fr}\, M)$, then

$$\mathbf{Prob}_\epsilon \left(\left| \lambda^\gamma(g) - \nu_{\mathrm{Fr}\, M}(g) \right| > \eta \right) \xrightarrow{\epsilon \to 0} 0.$$

Let $\mathbf{\Gamma}^{\geq \eta}_{\epsilon,R}$ be the $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$ so that $|\lambda^\gamma(g) - \nu_{\mathrm{Fr}\, M}(g)| \geq \eta$ and let $\mathbf{\Gamma}^{<\eta}_{\epsilon,R} := \mathbf{\Gamma}_{\epsilon,R} - \mathbf{\Gamma}^{\geq \eta}_{\epsilon,R}$. Then,

$$\begin{aligned} \left| \frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \lambda^\gamma(g) - \nu_{\mathrm{Fr}\, M}(g) \right| &\leq \frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}^{<\eta}_{\epsilon,R}} \left| \lambda^\gamma(g) - \nu_{\mathrm{Fr}\, M}(g) \right| + \sum_{\gamma \in \mathbf{\Gamma}^{\geq \eta}_{\epsilon,R}} \left| \lambda^\gamma(g) - \nu_{\mathrm{Fr}\, M}(g) \right| \\ &\leq \eta + 2\|g\|_{L^\infty(\mathrm{Fr}\, M)}\, \mathbf{Prob}_\epsilon(\mathbf{\Gamma}^{>\eta}_{\epsilon,R}). \end{aligned}$$

As $\eta > 0$ was arbitrary, this shows that

$$\lim_{\epsilon \to 0} \frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \lambda^{\gamma}(g) = \nu_{\mathrm{Fr}\, M}(g).$$

Finally, since g is continuous and $\delta \to 0$ as $\epsilon \to 0$, we also conclude that

$$\frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \lambda^{\gamma}(m_{\delta} g) \quad \text{and} \quad \frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \lambda^{\gamma}(M_{\delta} g)$$

converge to $\nu_{\mathrm{Fr}\, M}(g)$ as $\epsilon \to 0$, which concludes the proof of Lemma 5.3.3. $\square$

Proof of 5.3.4. To begin, for $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$ and for $g \in C(\mathrm{Fr}\, M)$, we define

$$\mathrm{Ft}_{\gamma}(g) := \sum_{\pi \in \mathbf{\Pi}_{\epsilon,R}(\gamma)} \Big(g(\mathbf{ft}^{\ell}\, \pi) + g(\mathbf{ft}^{r}\, \pi) \Big).$$

Using this notation,

$$\begin{aligned} \frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \phi^{\gamma}_{\epsilon,R}(g) - \phi_{\epsilon,R}(g) &= \frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \frac{\mathrm{Ft}_{\gamma}(g)}{C_{\epsilon,R,\gamma}} - \frac{1}{\#\widetilde{\mathbf{\Pi}}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \mathrm{Ft}_{\gamma}(g) \\ &= \frac{1}{\#\widetilde{\mathbf{\Pi}}_{\epsilon,R}} \sum_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \mathrm{Ft}_{\gamma}(g) \left(\frac{\#\widetilde{\mathbf{\Pi}}_{\epsilon,R} / \#\mathbf{\Gamma}_{\epsilon,R}}{C_{\epsilon,R,\gamma}} - 1 \right). \end{aligned}$$

Thus, we obtain

$$\left| \frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}} \phi^{\gamma}_{\epsilon,R}(g) - \phi_{\epsilon,R}(g) \right| \leq \frac{\#\mathbf{\Gamma}_{\epsilon,R}}{\#\widetilde{\mathbf{\Pi}}_{\epsilon,R}} \sup_{\gamma \in \mathbf{\Gamma}_{\epsilon,R}} \mathrm{Ft}_{\gamma}(g) \left(\frac{\#\widetilde{\mathbf{\Pi}}_{\epsilon,R} / \#\mathbf{\Gamma}_{\epsilon,R}}{C_{\epsilon,R,\gamma}} - 1 \right).$$

Using the fact that

$$|\,\mathrm{Ft}_{\gamma}(g)| \leq 2\#\mathbf{\Pi}_{\epsilon,R}(\gamma)\, \|g\|_{L^{\infty}(\mathrm{Fr}\, M)},$$

we have

$$\left|\frac{1}{\#\mathbf{\Gamma}_{\epsilon,R}}\phi^{\gamma}_{\epsilon,R}(g) - \phi_{\epsilon,R}(g)\right| \leq \|g\|_{L^\infty(\operatorname{Fr} M)} \sup_{\gamma\in\mathbf{\Gamma}_{\epsilon,R}}\left(\frac{2\#\mathbf{\Pi}_{\epsilon,R}(\gamma)}{C_{\epsilon,R,\gamma}} - \frac{2\#\mathbf{\Pi}_{\epsilon,R}(\gamma)}{\#\widetilde{\mathbf{\Pi}}_{\epsilon,R}/\#\mathbf{\Gamma}_{\epsilon,R}}\right). \tag{5.3.5}$$

The equidistribution of feet along a curve, Theorem 5.3.1, is what allows us to argue that the right hand side goes to zero as $\epsilon \to 0$. Say $a(\epsilon) \sim b(\epsilon)$ if $a(\epsilon)/b(\epsilon) \to 0$ as $\epsilon \to 0$. Then, Theorem 5.3.1 applied to $g = 1_{\operatorname{Fr} M}$ says that

$$1 - \delta \leq \frac{2\#\mathbf{\Pi}_{\epsilon,R}(\gamma)}{C_{\epsilon,R,\gamma}} \leq 1 + \delta,$$

where $\delta = e^{-qR}$, which implies

$$2\#\mathbf{\Pi}_{\epsilon,R}(\gamma) \sim C_{\epsilon,R,\gamma}$$

for any $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$, as well as

$$\frac{\#\widetilde{\mathbf{\Pi}}_{\epsilon,R}}{\#\mathbf{\Gamma}_{\epsilon,R}} \sim \frac{1}{\mathbf{\Gamma}_{\epsilon,R}}\sum_{\gamma\in\mathbf{\Gamma}_{\epsilon,R}} C_{\epsilon,R,\gamma}.$$

But from the definition of $C_{\epsilon,R,\gamma}$, we have

$$C_{\epsilon,R,\gamma} \sim \frac{1}{\mathbf{\Gamma}_{\epsilon,R}}\sum_{\#\gamma\in\mathbf{\Gamma}_{\epsilon,R}} C_{\epsilon,R,\gamma} \sim \tilde{c}_\epsilon \epsilon^4 e^{2R} R,$$

where $\tilde{c}_\epsilon$ is bounded in ϵ. Thus,

$$2\#\mathbf{\Pi}_{\epsilon,R}(\gamma) \sim C_{\epsilon,R,\gamma} \sim \frac{\#\widetilde{\mathbf{\Pi}}_{\epsilon,R}}{\#\mathbf{\Gamma}_{\epsilon,R}},$$

which allows us to conclude that the right hand side of 5.3.5 goes to zero as $\epsilon \to 0$. $\square$

To wrap up this section, we have proved Lemmas 5.3.3 and 5.3.4, which is what we needed to show that the feet of all pants equidistribute in $\operatorname{Fr} M$, namely

$$\phi_{\epsilon,R(\epsilon)} \xrightarrow{\star} \nu_{\operatorname{Fr} M}.$$

5.4 Approximate barycenters of pants

As before, we let v_R be the right action on $\operatorname{Fr} M \simeq \mathrm{PSL}_2\,\mathbf{C}$ of the element

$$a_{R/2}\, k\, a_{\log(\sqrt{3}/2)} \in \mathrm{PSL}_2\,\mathbf{C},$$

where $a_t = \operatorname{diag}(e^{t/2}, e^{-t/2})$ and $k \in \mathrm{SO}_2$ is the ninety-degree rotation bringing the first frame to the second.

We defined *left* and *right approximate barycenter* of an end of pants $\pi \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}$ respectively by

$$\mathbf{abar}^{\ell}(\pi) = v_R(\mathbf{ft}^{\ell}\,\pi) \quad \text{and} \quad \mathbf{abar}^{r}(\pi) = v_R(\mathbf{ft}^{r}\,\pi).$$

This section is dedicated to proving that the aproximate barycenters are indeed close to barycenters. Precisely,

Lemma 5.4.1. *Let $\pi \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}$ and $s = \ell$ or r. Then,*

$$\operatorname{dist}_{\operatorname{Fr} M}\left(\mathbf{abar}^{s}\,\pi, \mathbf{bar}^{s}\,\pi\right) \leq \omega(\epsilon),$$

where $\omega(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof. Throughout the proof, we will let $\omega(\epsilon)$ denote any quantity that goes to zero as $\epsilon \to 0$.

Let $\pi = [(f, C_i)] \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}$. Pick a representative (f, C_i) with geodesic cuffs and let $f(C_j) = \gamma_{j-i}$, for $i, j \in \mathbf{Z}/3$. Without loss of generality and to be explicit, we assume f to be

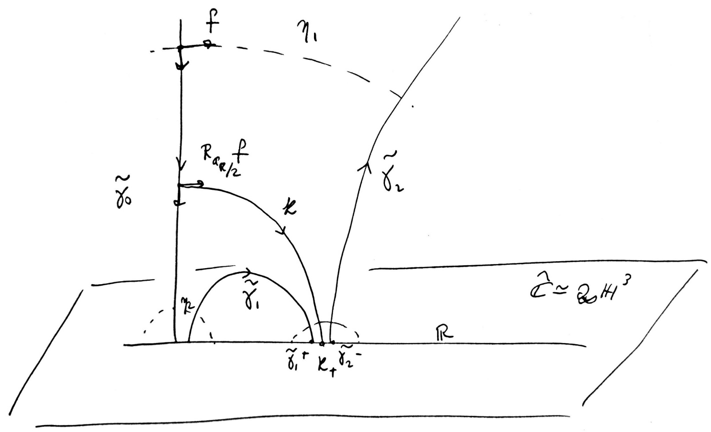

Figure 5.5: Lifting the pants π and its left foot to $\mathbf{H}^3$.

orientation-preserving.

Lift γ_0 to the geodesic $\tilde{\gamma}_0$ from ∞ to 0 in $\hat{\mathbf{C}} \simeq \partial_\infty \mathbf{H}^3$. Lift the left foot $\mathbf{ft}^\ell\,\pi$ to the frame based f at $e^{R/2}i$, whose first vector points at the direction of γ and whose second vector points at the positive direction of the real line $\mathbf{R} \subset \hat{\mathbf{C}}$.

Let κ be the ray given by

$$\kappa(t) := R_{a_t} R_k R_{a_{R/2}} f$$

for $t \geq 0$. The left approximate barycenter $\mathbf{abar}^\ell\,\pi$ lifts to the framed barycenter $\kappa(\log(\sqrt{3}/2))$ of the triangle with vertices $(\infty, 0, \kappa_+)$ associated to the side $(\infty, 0) = \tilde{\gamma}_0$.

Choose lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ of the other cuffs of π so they are connected to $\tilde{\gamma}_0$ by lifts of the short orthogeodesics, as in Figure 5.5. Note that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ lie, respectively, in geodesic planes $P_1$and P_2 with ∞ in their boundary that make an angle of $\omega(\epsilon)$ with each other and the plane P_0 that contains $\tilde{\gamma}_0$ and $(\kappa(t))_{t\geq 0}$.

The left triangle of π lifts to the triangle with vertices $(\infty, \tilde{\gamma}_1^-, \tilde{\gamma}_2^-)$ in this picture.

We let

$$\sigma(t) := R_{a_t} R_k R_{a_{\mathbf{hl}(\gamma)/2}} f,$$

for $t \geq 0$, be a lift of the orthogeodesic ray from γ_0 to itself. Then, σ^+ lies in the annulus E of $\hat{\mathbf{C}}$ so that $\tilde{\gamma}_1^+, \tilde{\gamma}_2^- \in \partial E$. Since the geodesic planes P_1 and P_2 contianing $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ make a small angle with each other, it follows that σ^+ is within distance $\omega(\epsilon)$ in $\hat{\mathbf{C}}$ of $\tilde{\gamma}_1^+$ and $\tilde{\gamma}_2^-$.

On the other hand, since $R_{a_{R/2}} f$ and $R_{a_{\mathbf{hl}(\gamma)/2}} f$ are at a distance $O(\epsilon)$ of each other in $\operatorname{Fr} \mathbf{H}^3$, it follows that σ^+ and κ^+ are at a distance $O(\epsilon)$ in $\hat{\mathbf{C}}$.

We conclude that the vertices $(\infty, \tilde{\gamma}_1^-, \tilde{\gamma}_2^-)$ of the left triangle of π are within distance $\omega(\epsilon)$ of the vertices $(\infty, 0, \kappa^+)$ of the triangle whose framed barycenter associated to $(\infty, 0)$ is $\mathbf{abar}^\ell\, \pi$. This means that all the framed barycenters of these triangles are within $\omega(\epsilon)$ of each other in $\operatorname{Fr} \mathbf{H}^3$. Thus,

$$\operatorname{dist}_{\operatorname{Fr} M}(\mathbf{abar}^\ell\, \pi, \mathbf{bar}^\ell\, \pi) \leq \omega(\epsilon),$$

as desired.

The proof follows in the same way for the right barycenters. □

5.5 Equidistribution of approximate barycenters in $\operatorname{Fr} M$

In this section, we will show that the probability uniform measure $\beta^a_{\epsilon,R}$ supported on the approximate barycenters of $\pi \in \widetilde{\mathbf{\Pi}}_{\epsilon,R}$ equidistributes as $\epsilon \to 0$ and $R(\epsilon) \to \infty$.

We first claim

Lemma 5.5.1. *As $\epsilon \to 0$ and $R(\epsilon) \to \infty$,*

$$(R_{a_{R/2}})_* \phi_{\epsilon,R} \xrightarrow{\star} \nu_{\operatorname{Fr} M}.$$

Proof. Let $g \in C(\operatorname{Fr} M)$ be nonnegative. Along each $\gamma \in \mathbf{\Gamma}_{\epsilon,R}$, the Lebesgue probability measure λ^γ is invariant under the right action of $a_t \in \operatorname{PSL}_2 \mathbf{C}$. Therefore, the equidistribution of feet along a curve, Lemma 5.3.1 gives us

$$(1-\delta)\lambda^{\gamma}(m_\delta g) \leq (R_{a_{R/2}})_* \phi^{\gamma}_{\epsilon,R}(g) \leq (1+\delta)\lambda^{\gamma}(M_\delta g),$$

where as usual $\delta = e^{-qR}$. Thus, the arguments from section 5.2 show that

$$(R_{a_{R/2}})_* \phi^{\gamma}_{\epsilon,R}(g) \to \nu_{\operatorname{Fr} M}(g)$$

as $\epsilon \to 0$ and $R(\epsilon) \to \infty$. □

We also observe

Lemma 5.5.2. *Suppose ν_i are probability measures in $\operatorname{Fr} M$ so that $\nu_i \xrightarrow{\star} \nu_{\operatorname{Fr} M}$ as $i \to \infty$. Then, given $h \in \mathrm{PSL}_2\,\mathbf{C}$, we have*

$$(R_h)_* \nu_i \xrightarrow{\star} \nu_{\operatorname{Fr} M}$$

Proof. Suppose $g \in C(\operatorname{Fr} M)$. Then,

$$(R_h)_* \nu_i(g) = \nu_i(g \circ R_h^{-1}).$$

Thus $(R_h)_* \nu_i(g) \to \nu_{\operatorname{Fr} M}(g \circ R_h^{-1})$ as $i \to \infty$. As $\nu_{\operatorname{Fr} M}$ is invariant under the right action of $\mathrm{PSL}_2\,\mathbf{C}$, we conclude. □

By construction,

$$\beta^a_{\epsilon,R} = (R_{k a_{\log(\sqrt{3}/2)}})_* (R_{a_{R/2}})_* \phi_{\epsilon,R}.$$

Combining the two lemmas above, we conclude that $\beta^a_{\epsilon,R} \xrightarrow{\star} \nu_{\operatorname{Fr} M}$ as $\epsilon \to 0$ and $R(\epsilon) \to \infty$.

5.6 Conclusion: equidistribution of barycenters in $\operatorname{Fr} M$

Finally, as a corollary of the equidistribution of the approximate barycenters, we obtain the equidistribution of the actual barycenters, which is the main theorem of the chapter. For $g \in C(\operatorname{Fr} M)$, we have

$$\beta^a_{\epsilon,R}(g) - \beta_{\epsilon,R}(g) = \frac{1}{2\#\widetilde{\mathbf{\Pi}}_{\epsilon,R}} \sum_{\pi \in t\mathbf{\Pi}_{\epsilon,R}} \sum_{s\in\{\ell,r\}} (g(\mathbf{abar}^s\,\pi) - g(\mathbf{bar}^s\,\pi))$$

As g is uniformly continuous and

$$\operatorname{dist}_{\operatorname{Fr} M}(\mathbf{abar}^s\,\pi, \mathbf{bar}^s\,\pi) \leq \omega(\epsilon),$$

where $\omega(\epsilon) \to 0$ as $\epsilon \to 0$, we conclude that

$$|\beta^a_{\epsilon,R}(g) - \beta_{\epsilon,R}(g)| \xrightarrow{\epsilon\to 0} 0.$$

Thus, since $\beta^a_{\epsilon,R}(g) \to \nu_{\operatorname{Fr} M}(g)$, we conlcude that $\beta_{\epsilon,R}(g) \to \nu_{\operatorname{Fr} M}(g)$ as $\epsilon \to 0$ and $R(\epsilon) \to \infty$.

Chapter 6

Equidistributing surfaces

Let $S_{\epsilon,R}$ be the connected, closed, π_1-injective and $(1 + O(\epsilon))$-quasifuchsian surface made out of $N(\epsilon, R)$ copies of each pants in $\mathbf{\Pi}_{\epsilon,R}$, as explained in Chapter 2. Let $\nu_{S_{\epsilon,R}}$ be their probability area measures on $\operatorname{Gr} M$. Note that these measures are also the probability area measure of the possibly disconnected surface built out of one copy of each $\pi \in \mathbf{\Pi}_{\epsilon,R}$. Using the fact that the barycenters of good pants are well distributed, we will show

Theorem 6.0.1. *As $\epsilon \to 0$ and $R(\epsilon) \to \infty$ fast enough,*

$$\nu_{S_{\epsilon,R(\epsilon)}} \overset{\star}{\rightharpoonup} \nu_{\operatorname{Gr} M},$$

where $\nu_{\operatorname{Gr} M}$ is the probability volume measure of $\operatorname{Gr} M$*.*

Outside of the pleating lamination, we may define the unit tangent bundle $\mathrm{T}^1 S_{\epsilon,R}$ of $S_{\epsilon,R}$. This can be seen as a three-dimensional submanifold of $\operatorname{Fr} M$, where $(p, v) \in \mathrm{T}^1 S_{\epsilon,R}$ is included in $\operatorname{Fr} M$ as the frame $(p, v, w, v \times w)$, where w is the image of v under the ninety-degree rotation $k \in \mathrm{SO}_2$ described in the last chapter.

Let $\nu_{\mathrm{T}^1 S_{\epsilon,R}}$ be the probability volume measure of $\mathrm{T}^1 S_{\epsilon,R}$ on $\operatorname{Fr} M$. We show

Claim 6.0.2. *As $\epsilon \to 0$ and $R(\epsilon) \to \infty$,*

$$\nu_{\mathrm{T}^1 S_{\epsilon,R(\epsilon)}} \overset{\star}{\rightharpoonup} \nu_{\mathrm{Fr}\, M}.$$

Proof. This proof is similar to pages 23-26 of [12].

Let $\Delta \subset \mathrm{PSL}_2\,\mathbf{R}$ be the set so that $R_\Delta(b)$ is the unit tangent bundle of the ideal triangle in M with $b \in \mathrm{Fr}\, M$ as a framed barycenter (for any $b \in \mathrm{Fr}\, M$). Let $\nu_{\mathrm{PSL}_2\,\mathbf{R}}$ be the probability Haar measure on $\mathrm{PSL}_2\,\mathbf{R}$.

Thus, given $g \in C(\mathrm{Fr}\, M)$,

$$\nu_{\mathrm{T}^1 S_{\epsilon,R}}(g) = \int_{\mathrm{Fr}\, M} \frac{1}{\nu_{\mathrm{PSL}_2\,\mathbf{R}}(\Delta)} \int_\Delta g(R_t^{-1} b)\, d\nu_{\mathrm{PSL}_2\,\mathbf{R}}(t)\, d\beta_{\epsilon,R}.$$

By Fubini's theorem,

$$\nu_{\mathrm{T}^1 S_{\epsilon,R}}(g) = \frac{1}{\nu_{\mathrm{PSL}_2\,\mathbf{R}}(\Delta)} \int_\Delta \beta_{\epsilon,R}(g \circ R_t^{-1})\, d\nu_{\mathrm{PSL}_2\,\mathbf{R}}(t).$$

From the equidistribution of the barycenters and the $\mathrm{PSL}_2\,\mathbf{C}$-invariance of $\nu_{\mathrm{Fr}\, M}$, the integrand $\beta_{\epsilon,R}(g \circ R_t^{-1})$ converges to $\nu_{\mathrm{Fr}\, M}(g)$. Using the dominated convergence theorem, we conclude that

$$\nu_{\mathrm{T}^1 S_{\epsilon,R}}(g) \xrightarrow{\epsilon \to 0} \frac{1}{\nu_{\mathrm{PSL}_2\,\mathbf{R}}(\Delta)} \int_\Delta \nu_{\mathrm{Fr}\, M}(g)\, d\nu_{\mathrm{PSL}_2\,\mathbf{R}} = \nu_{\mathrm{Fr}\, M}(g).$$

$\square$

For $g \in C(\mathrm{Fr}\, M)$, we let $\tilde{g} \in C(\mathrm{Gr}\, M)$ be the function defined by

$$\tilde{g}(p, P) = \frac{1}{2\pi} \int_0^{2\pi} g(p, r_\theta f)\, d\theta,$$

where $r_\theta \in \mathrm{PSL}_2\,\mathbf{R}$ is the rotation of degree θ and f is the frame whose first two vectors span the oriented plane P.

Fubini's theorem tells us that for $g \in C(\mathrm{Fr}\,M)$,

$$\nu_{\mathrm{T}^1\,S_{\epsilon,R}}(g) = \nu_{S_{\epsilon,R}}(\tilde{g}).$$

Moreover, any $h \in C(\mathrm{Gr}\,M)$ is of the form $h = \tilde{g}$, where $h(p, P) = g(p, f)$ for any frame f whose first two vectors span P.

Thus Claim 6.2 implies Theorem 6.1, and we can conclude that the connected surfaces made out of $N(\epsilon, R, M)$ copies of each pants in $\mathbf{\Pi}_{\epsilon,R}$ equidistribute in $\mathrm{Gr}\,M$ as $\epsilon \to 0$ and $R(\epsilon) \to \infty$.

Chapter 7

Non-equidistributing surfaces

Let $\mathscr{G}$ be a set containing a representative of each commensurability class of closed immersed totally geodesic surfaces in M. Let $(\mathscr{G}_k)_{k\geq 1} \subseteq \mathscr{G}$ be an increasing sequence of finite subsets, so that $\bigcup_{k\geq 1} \mathscr{G}_k = \mathscr{G}$. (In the case when $\mathscr{G}$ is finite, it suffices to take $\mathscr{G}_k = \mathscr{G}$ for all $k \geq 1$.) Kahn and Marković [9] proved that given $k \geq 1$, $\epsilon > 0$ small enough and $R = R(\epsilon, k)$ large enough, each $T \in \mathscr{G}_k$ has a finite cover $\hat{T}$ which admits a pants decomposition of pants in $\mathbf{\Pi}_{\epsilon,R}$ that are all glued via (ϵ, R)-good gluings. (This fact was used to prove the Ehrenpreis conjecture.) By possibly passing to a further double cover, we can assume the cuffs of each $\hat{T}$ are all nonseparating, as explained in the proof of Theorem 4.0.2.

For each $T \in \mathscr{G}_k$, let T^d be a cover of $\hat{T}$ of degree $d = d(T, \epsilon, R)$. We may choose this cover so that T^d also admits a pants decomposition, denoted Π_T, by pants in $\mathbf{\Pi}_{\epsilon,R}$ that are glued via (ϵ, R)-good gluings.

Let $\hat{S}_{\epsilon,R}$ be the connected, closed, π_1-injective and $(1 + O(\epsilon))$-quasifuchsian surface produced in the previous chapter. We may assume that $\hat{S}_{\epsilon,R}$ is built out of $N(\epsilon, R, k) \geq \#\mathscr{G}_k$ copies of each $\pi \in \mathbf{\Pi}_{\epsilon,R}$.

For each $T \in \mathscr{G}_k$, choose a curve $\gamma \subset T^d$ that arises as a boundary of a pants in Π_T.

Let $\pi_T^- \in \mathbf{\Pi}^-_{\epsilon,R}(\gamma_T)$ and $\pi_T^+ \in \mathbf{\Pi}^+_{\epsilon,R}(\gamma_T)$ be pants in Π_T that are (ϵ, R)-well glued along γ_T. Namely,

$$\left|\mathbf{ft}\,\pi_T^- - \tau\left(\mathbf{ft}\,\pi_T^+\right)\right| < \frac{\epsilon}{R},$$

where as before $\tau(x) = x + 1 + i\pi$.

As argued in the proof of Theorem 4.0.2, since $\hat{S}_{\epsilon,R}$ is built out of N copies of *each* $\pi \in \mathbf{\Pi}_{\epsilon,R}$, there is a pants $p_T^- \in \mathbf{\Pi}^-_{\epsilon,R}(\gamma_T)$ in $\hat{S}_{\epsilon,R}$ so that

$$|\,\mathbf{ft}\,\pi_T^- - \mathbf{ft}\,p_T^-| < \frac{\epsilon}{R}.$$

On the other hand, p_T^- is (ϵ, R)-well glued to a pants p_T^+ also from $\hat{S}_{\epsilon,R}$, i.e.,

$$\left|\mathbf{ft}\,p_T^- - \tau\left(\mathbf{ft}\,p_T^-\right)\right| < \frac{\epsilon}{R}.$$

Putting these inequalities together, we have that

$$\left|\mathbf{ft}\,p_T^- - \tau\left(\mathbf{ft}\,\pi_T^+\right)\right| < 2\frac{\epsilon}{R} \quad \text{and} \quad \left|\mathbf{ft}\,\pi_T^- - \tau\left(\mathbf{ft}\,p_T^+\right)\right| < 2\frac{\epsilon}{R}.$$

In other words, we may cut along γ_T and reglue p_T^- to π_T^+ and π_T^- to p_T^+ in a $(2\epsilon, R)$-good way. We call this reglued surface $S_{\epsilon,R,\mathbf{d}}$, where $\mathbf{d} = (d(T, \epsilon, R))_{T\in\mathscr{G}_k}$ is a vector keeping track of the degrees of each cover $T^d \to \hat{T}$.

The regluings are done along nonseparating curves, so each $S_{\epsilon,R,\mathbf{d}}$ is closed, oriented, *connected* and $(1 + O(\epsilon))$-quasifuchsian. (The connectedness uses the fact that the regluings were done along nonseparating cuffs.) As usual, we let $\nu(S_{\epsilon,R,\mathbf{d}})$ denote the probability area measure of $S_{\epsilon,R,\mathbf{d}}$ on the Grassmann bundle $\mathrm{Gr}\,M$. Recall that $\nu_{\mathrm{Gr}\,M}$ denotes the Haar measure on $\mathrm{Gr}\,M$ and ν_T denotes the area measure of T on $\mathrm{Gr}\,M$.

Proposition 7.0.1. *The weak-* limit points, as $\epsilon \to 0$ and $k \to \infty$, of the measures $\nu(S_{\epsilon,R(\epsilon,k),\mathbf{d}(\epsilon,k)})$*

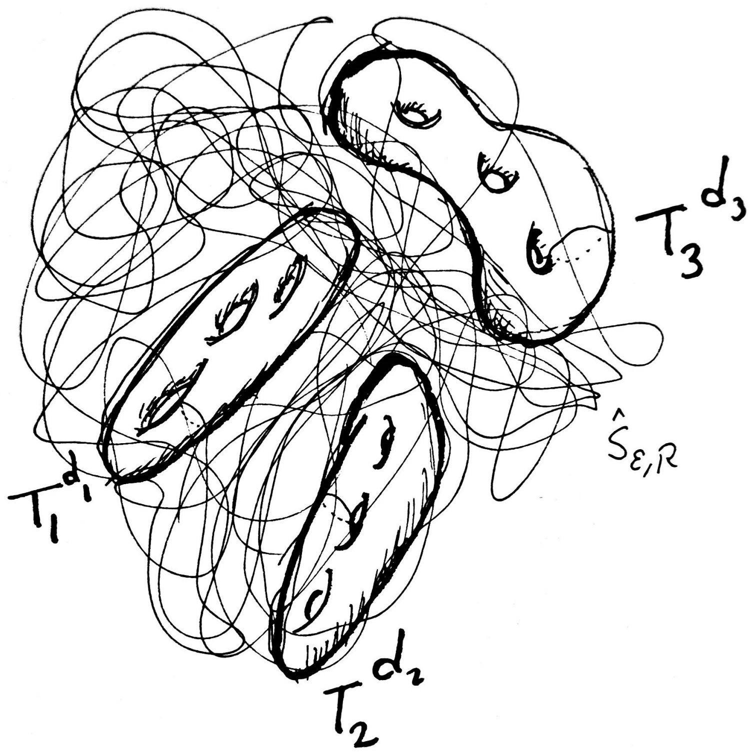

Figure 7.1: The family of surfaces $S_{\epsilon,R,\mathbf{d}}$, which can accumulate on the totally geodesic surfaces T by appropriately choosing the degrees d of their covers.

on Gr M *consist of all measures* ν *of the form*

$$\nu = \alpha_M \nu_{\operatorname{Gr} M} + \sum_{T \in \mathscr{G}} \alpha_T \nu_T. \tag{$\star$}$$

Proof. Let g_T denote the genus of a totally geodesic surface T and let $g_{\epsilon,R}$ denote the genus of $\hat{S}_{\epsilon,R}$.

We can write the area measure $\nu(S_{\epsilon,R,\mathbf{d}})$ as

$$\nu(S_{\epsilon,R,\mathbf{d}}) = \frac{1}{\sum_{T \in \mathscr{G}_k} 2\pi(g_T - 1)d(T) + 2\pi(g_{\epsilon,R} - 1)} \left(\sum_{T \in \mathscr{G}_k} 2\pi(g_T - 1)d(T)\nu_T + 2\pi(g_{\epsilon,R} - 1)\nu(\hat{S}_{\epsilon,R}) \right).$$

Recall that $\nu(\hat{S}_{\epsilon,R(\epsilon)}) \xrightarrow{\star} \nu_{\operatorname{Gr} M}$ as $\epsilon \to 0$. Thus, by making $d(T, \epsilon, R(\epsilon, k))$ grow appropriately fast for each T, we can make $\nu(S_{\epsilon,R,\mathbf{d}})$ converge to any given measure of the form ($\star$) as $\epsilon \to 0$ and $k \to \infty$. $\square$

This, together with Theorem 1.2, which was proved in Chapter 4, completes the proof of Theorem 1.1.

Bibliography

[1] Bader, Uri; Fisher, David; Miller, Nicholas; Stover, Matthew Arithmeticity, superrigidity, and totally geodesic submanifolds. *Ann. of Math.* (2) 193 (2021), no. 3, 837–861.

[2] Calegari, Danny; Marques, Fernando C.; Neves, André. Counting minimal surfaces in negatively curved 3-manifolds. Arxiv preprint, arXiv:2002.01062 [math.DG].

[3] Epstein, D. B. A., Marden A. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, 113-253, London Math. Soc. Lecture Note Ser., 111, Cambridge Univ. Press, Cambridge, 1987.

[4] Gardiner, Frederick P. *Teichmüller theory and quadratic differentials.* Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1987.

[5] Hamenstädt, Ursula. Incompressible surfaces in rank one locally symmetric spaces. *Geom. Funct. Anal.* 25 (2015), no. 3, 815–859.

[6] Howards, Hugh; Hutchings, Michael; Morgan, Frank. The isoperimetric problem on surfaces, *The American Mathematical Monthly*, Vol. 106, No. 5 (May, 1999), pp.430-439.

[7] Kahn, Jeremy; Labourie, François; Mozes, Shahar. Surface subgroups in uniform lattices of some semi-simple groups. arXiv preprint, arxiv:1805.10189.

[8] Kahn, Jeremy; Marković, Vladimir. Immersing almost geodesic surfaces in a closed hyperbolic three manifold. *Ann. of Math. (2)* 175 (2012), no. 3, 1127–1190.

[9] Kahn, Jeremy; Marković, Vladimir. The good pants homology and the Ehrenpreis Conjecture. *Ann. of Math.* 182 (2015), no. 1, 1-72.

[10] Kahn, Jeremy; Wright, Alex. Nearly Fuchsian surface subgroups of finite covolume Kleinian groups. ArXiv preprint, arXiv:1809.07211 [math.GT].

[11] Kahn, Jeremy; Wright, Alex. Counting connections in a locally symmetric space. http://www.math.brown.edu/jk17/Connections.pdf.

[12] Labourie, François. Asymptotic counting of minimal surfaces in hyperbolic 3-manifolds [according to Calegari, Marques and Neves]. arXiv:2203.09366v1 [math.DG].

[13] Lalley, Steve. Distribution of periodic orbits of symbolic and Axiom A flows. *Adv. in Appl. Math. 8* (1987), no. 2, 154–193.

[14] Liu, Yi; Marković, Vladimir. Homology of curves and surfaces in closed hyperbolic 3-manifolds. *Duke Math. J.* 164 (2015), no. 14, 2723–2808.

[15] Lowe, Ben. Deformations of totally geodesic foliations and minimal surfaces in negatively curved 3-manifolds. Geom. Funct. Anal. 31 (2021), no. 4, 895–929.

[16] Maclachlan, C.; Reid, A. W. Commensurability classes of arithmetic Kleinian groups and their Fuchsian subgroups. *Math. Proc. Cambridge Philos. Soc.* 102 (1987), no. 2, 251–257.

[17] Mohammadi, Amir; Margulis, Gregorii Arithmeticity of hyperbolic 3-manifolds containing infinitely many totally geodesic surfaces. *Ergodic Theory Dynam. Systems* 42 (2022), no. 3, 1188–1219.

[18] Ratner, Marina. On Raghunathan's measure conjecture. *Ann. of Math. (2)* 134 (1991), no. 3, 545-607.

[19] Reid, Alan W. Totally geodesic surfaces in hyperbolic 3-manifolds. *Proc. Edinburgh Math. Soc.* (2) 34 (1991), no. 1, 77–88.

[20] Sacks, J.; Uhlenbeck, K. Minimal immersions of closed Riemann surfaces. *Trans. Amer. Math. Soc.* 271 (1982), no. 2, 639–652. (Reviewer: R. Osserman)

[21] Seppi, Andrea. Minimal discs in hyperbolic space bounded by a quasicircle at infinity. *Comment. Math. Helv.* 91 (2016), no. 4, 807-839.

[22] Mozes, Shahar; Shah, Nimish. On the space of ergodic invariant measures of unipotent flows. *Ergodic Theory Dynam. Systems* 15 (1995), no. 1, 149–159.

[23] Schoen, R.; Yau, Shing Tung Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. *Ann. of Math.* (2) 110 (1979), no. 1, 127–142. (Reviewer: Jonathan Sacks)

[24] Sun, Hongbin The panted cobordism groups of cusped hyperbolic 3-manifolds. J. Topol. 15 (2022), no. 3, 1580–1634. (Reviewer: Ken'ichi Yoshida)

[25] Thurston, William P. *The Geometry and Topology of Three-Manifolds.* Princeton lecture notes.

[26] Uhlenbeck, Karen K. Closed minimal surfaces in hyperbolic 3-manifolds. *Seminar on minimal submanifolds,* 147-168, *Ann. of Math. Stud.*, 103, Princeton Univ. Press, Princeton, NJ, 1983.